ST 430: Homework #3 Solutions

5.6.2 (a) $F_Y(y) = 1 - e^{-(y-\theta)}$, $\theta \le y$, so

$f_{Y_{\min}}(y) = nf_Y(y)[1 - F_Y(y)]^{n-1} = ne^{-(y-\theta)}[e^{-(y-\theta)}]^{n-1} = ne^{-n(y-\theta)}$

$$\prod_{i=1}^{n} e^{-(y_i-\theta)} = e^{-\sum_{i=1}^{n} y_i} e^{n\theta}) = ne^{-n(y_{\min}-\theta)}\left[\frac{1}{n}e^{ny_{\min}}e^{-\sum_{i=1}^{n} y_i}\right].$$ Thus, $Y_{\min}$ is sufficient by Theorem 5.6.1.

(b) We need to show that the likelihood function given $y_{\max}$ is independent of θ. But the likelihood function is

$$\prod_{i=1}^{n} e^{-(y_i-\theta)} = \begin{cases} e^{\theta}e^{-\sum_{i=1}^{n} y_i} \text{ if } \theta \le y_1, y_2, \ldots y_n \\ 0 \text{ otherwise} \end{cases}$$

Regardless of the value of $y_{\max}$, the expression for the likelihood does depend on $\tilde{\theta}$. If any one of the y_i, other than $y_{\max}$, is less than θ, the expression is 0. Otherwise it is non-zero.

5.6.5 $$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{y_i^2}{\sigma^2}} = \left[(\sigma^2)^{-n/2} e^{-\frac{1}{2}\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} y_i^2\right)}\right][2\pi^{-n/2}],$$ so $\sum_{i=1}^{n} Y_i^2$ is sufficient by Theorem 5.6.1.

5.6.6 $$\prod_{i=1}^{n} \frac{1}{(r-1)!\theta^r} y_i^{r-1} e^{-y_i/\theta} = \frac{1}{[(r-1!]^n} \frac{1}{\theta^{rn}} \left(\prod_{i=1}^{n} y_i\right)^{r-1} = \left[\frac{1}{\theta^{rn}} e^{-\frac{1}{\theta}\sum_{i=1}^{n} y_i}\right] \frac{1}{[(r-1)!]^n} \left(\prod_{i=1}^{n} y_i\right)^{r-1}$$

so $\sum_{i=1}^{n} Y_i$ is a sufficient statistic for θ. So also is $\frac{1}{r}\overline{Y}$. (See Question 5.6.4)

5.6.9 $\lambda e^{-\lambda y} = e^{\ln \lambda - \lambda y} = e^{y(-\lambda) + \ln \lambda}$. Take $K(y) = y$, $p(\lambda) = -\lambda$, $S(y) = 0$, and $q(\lambda) = \ln \lambda$.

Then $\sum_{i=1}^{n} Y_i$ is sufficient.

5.6.10 $\theta/(1+y)^{\theta+1} = e^{\ln \theta - (\theta+1)\ln(1+y)} = e^{[\ln(1+y)](-\theta-1) + \ln \theta}$. Take $K(y) = \ln(1+y)$, $p(\theta) = -\theta - 1$, and $q(\theta) = \ln \theta$. Then $\sum_{i=1}^{n} K(Y_i) = \sum_{i=1}^{n} \ln(1+Y_i)$ is sufficient for θ.