ST 430: Homework #4 Solutions

5.7.2 Since $\mu = 0$, for each i, $E\left(Y_i^2\right) = \sigma^2$. By the weak law of large numbers demonstrated in Example 5.7.2, $\frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is a consistent estimator of the mean of the Y_i^2, in this case σ^2. However, the proof given in the example requires that $\text{Var}\left(Y_i^2\right) < \infty$. This follows from an application of the moment generating function for the normal distribution.

5.7.4 a) Let $\mu_n = E(\hat{\theta}_n)$. $E\left[(\hat{\theta}_n - \theta)^2\right] = E\left[(\hat{\theta}_n - \mu_n + \mu_n - \theta)^2\right]$

$= E\left[(\hat{\theta}_n - \mu_n)^2 + (\mu_n - \theta)^2 + 2(\hat{\theta}_n - \mu_n)(\mu_n - \theta)\right]$

$= E[(\hat{\theta}_n - \mu_n)^2] + E[(\mu_n - \theta)^2] + 2(\mu_n - \theta)E[(\hat{\theta}_n - \mu_n)]$

$= E\left[(\hat{\theta}_n - \mu_n)^2\right] + (\mu_n - \theta)^2 + 0$

or $E\left[(\hat{\theta}_n - \theta)^2\right] = E\left[(\hat{\theta}_n - \mu_n)^2\right] + (\mu_n - \theta)^2$

The left hand side of the equation tends to 0 by the squared-error consistency hypothesis. Since the two summands on the right hand side are non-negative, each of them must tend to zero also. Thus,

$\lim_{n\to\infty} (\mu_n - \theta)^2 = 0$, which implies $\lim_{n\to\infty} (\mu_n - \theta) = 0$, or $\lim_{n\to\infty} \mu_n = \theta$.

b) By Part (a) $\lim_{n\to\infty} \mu_n - \theta = 0$. For any $\varepsilon > 0$,

$\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| \geq \varepsilon\right) = \lim_{n\to\infty} P\left(\left|(\hat{\theta}_n - \mu_n) - (\mu_n - \theta)\right| \geq \varepsilon\right)$

$= \lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \mu_n\right| \geq \varepsilon\right) \leq \frac{\text{Var}(\hat{\theta}_n)}{\varepsilon^2}$ by Chebyshev's Inequality.

$\frac{\text{Var}(\hat{\theta}_n)}{\varepsilon^2} = \frac{E[(\hat{\theta}_n - \mu_n)^2]}{\varepsilon^2}$, and by Part (a), $\lim_{n\to\infty} E\left[(\hat{\theta}_n - \mu_n)^2\right] = 0$,

so $\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| \geq \varepsilon\right) = 0$. Thus, $\hat{\theta}_n$ is consistent.

5.7.5 $E\left[(Y_{\max} - \theta)^2\right] = \int_0^\theta (y - \theta)^2 \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1} dy$

$= \frac{n}{\theta^n}\int_0^\theta (y^{n+1} - 2\theta y^n + \theta^2 y^{n-1})dy = \frac{n}{\theta^n}\left(\frac{\theta^{n+2}}{n+2} - \frac{2\theta^{n+2}}{n+1} + \frac{\theta^{n+2}}{n}\right)$

$= \left(\frac{n}{n+2} - \frac{2n}{n+1} + 1\right)\theta^2$

Then $\lim_{n\to\infty} E\left[(Y_{\max} - \theta)^2\right] = \lim_{n\to\infty}\left(\frac{n}{n+2} - \frac{2n}{n+1} + 1\right)\theta^2 = 0$ and the estimator is squared error consistent.

5.3.2 The confidence interval is $\left(\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) =$

$\left(70.833 - 1.96\frac{8.0}{\sqrt{6}}, 70.833 + 1.96\frac{8.0}{\sqrt{6}}\right)$ = (64.432, 77.234). Since 80 does not fall within the confidence interval, that men and women metabolize methylmercury at the same rate is not believable.

5.3.4
a) $P(-1.64 < Z < 2.33) = 0.94$, a 94% confidence level.
b) $P(-\infty < Z < 2.58) = 0.995$, a 99.5% confidence level.
a) $P(-1.64 < Z < 0) = 0.45$, a 45% confidence level.

5.3.8 From Theorem 5.3.1, the confidence interval is

$$\left(\frac{179}{220} - 1.64\sqrt{\frac{(179/220)(1-179/220)}{220}},\quad \frac{179}{220} + 1.64\sqrt{\frac{(179/220)(1-179/220)}{220}}\right)$$

= (0.771, 0.857)

5.3.10 Let p be the probability that a viewer would watch less than a quarter of the advertisements during Super Bowl XXIX. The confidence interval for p is

$$\left(\frac{281}{1015} - 1.64\sqrt{\frac{(281/1015)(1-281/1015)}{1015}},\quad \frac{281}{1015} + 1.64\sqrt{\frac{(281/1015)(1-281/1015)}{1015}}\right)$$

= (0.254, 0.300)

5.3.12
$$\frac{x}{n} - 0.67\sqrt{\frac{(x/n)(1-x/n)}{n}} = 0.57$$
$$\frac{x}{n} + 0.67\sqrt{\frac{(x/n)(1-x/n)}{n}} = 0.63$$

Adding the two equations gives $2\frac{x}{n} = 1.20$ or $\frac{x}{n} = 0.60$

Substituting the value for $\frac{x}{n}$ into the first equation above gives

$$0.60 - 0.67\sqrt{\frac{(0.60)(1-0.60)}{n}} = 0.57.$$

Solving this equation for n gives $n = 120$.

5.3.16 $g(p) = p - p^2$. $g'(p) = 1 - 2p$. Setting $g'(p) = 0$ gives $p = 1/2$.
$g''(p) = -2$. Since the second derivative is negative at $p = 1/2$, a maximum occurs there. The maximum value of $g(p)$ is $g(1/2) = 1/4$.