ST 430
Homework #6: Solutions

6.4.6 a) In order for α to be 0.07, $P(60 - \bar{y}^* \le \bar{Y} \le 60 + \bar{y}^* \mid \mu = 60) = 0.07$. Equivalently,

$$P\left(\frac{60-\bar{y}^*-60}{8.0/\sqrt{36}} \le \frac{\bar{Y}-60}{8.0/\sqrt{36}} \le \frac{60+\bar{y}^*-60}{8.0/\sqrt{36}}\right) = P(-0.75\bar{y}^* \le Z \le 0.75\bar{y}^*) = 0.07.$$ But

$P(-0.09 \le Z \le 0.09) = 0.07$, so $0.75\bar{y}^* = 0.09$, which implies that $\bar{y}^* = 0.12$.

b) $1 - \beta = P(\text{reject } H_0 \mid H_1 \text{ is true}) = P(59.88 \le \bar{Y} \le 60.12 \mid \mu = 62) =$

$$P\left(\frac{59.88-62}{8.0/\sqrt{36}} \le Z \le \frac{60.12-62}{8.0/\sqrt{36}}\right) = P(-1.59 \le Z \le -1.41) = 0.0793 - 0.0559 =$$

0.0234.

c) For $\alpha = 0.07$, $\pm z_{\alpha/2} = \pm 1.81$ and H_0 should be rejected if $\bar{y}$ is either

1) $\le 60 - 1.81 \cdot \frac{8.0}{\sqrt{36}} = 57.50$ or **2)** $\ge 60 + 1.81 \cdot \frac{8.0}{\sqrt{36}} = 62.41$. Suppose $\mu = 62$. Then

$1 - \beta = P(\bar{Y} \le 57.59 \mid \mu = 62) + P(\bar{Y} \ge 62.41 \mid \mu = 62) = P(Z \le -3.31) + P(Z \ge 0.31) =$
$0.0005 + 0.3783 = 0.3788$.

6.4.8 If $n = 45$, H_0 will be rejected when $\bar{y}$ is either **1)** $\le 10 - 1.96 \cdot \frac{4}{\sqrt{45}} = 8.83$ or **2)** $\ge 10 +$ $1.96 \cdot \frac{4}{\sqrt{45}} = 11.17$. When $\mu = 12$, $\beta = P(\text{accept } H_o \mid H_1 \text{ is true}) = P(8.83 \le \bar{Y} \le 11.17 \mid \mu =$ $12) = P\left(\frac{8.83-12}{4/\sqrt{45}} \le Z \le \frac{11.17-12}{4/\sqrt{45}}\right) = P(-5.32 \le Z \le -1.39) = 0.0823$. It follows that a sample of size $n = 45$ <u>is</u> sufficient to keep β smaller than 0.20 when $\mu = 12$.

6.4.10 a) $P(\text{Type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y \ge 3.20 \mid \lambda = 1) =$

$$\int_{3.20}^{\infty} e^{-y}\,dy = 0.04.$$

b) $P(\text{Type II error}) = P(\text{accept } H_0 \mid H_1 \text{ is true}) =$

$$P\left(Y < 3.20 \middle| \lambda = \frac{4}{3}\right) = \int_0^{3.20} \frac{3}{4} e^{-3y/4}\,dy = \int_0^{2.4} e^{-u}\,du = 0.91.$$

6.4.16 If H_0 is true, $X = X_1 + X_2$ has a binomial distribution with $n = 6$ and $p = \frac{1}{2}$. Therefore,

$$\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P\left(X \ge 5 \middle| p = \frac{1}{2}\right) = \sum_{k=5}^{6} \binom{6}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{6-k} = 7/2^6 = 0.11.$$

6.4.18 a) $\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(X \le 2 \mid \lambda = 6) = \sum_{k=0}^{2} \frac{e^{-6}6^k}{k!} = 0.062$.

b) $\beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(X \ge 3 \mid \lambda = 4) = 1 - P(X \le 2 \mid \lambda = 4) =$
$1 - \sum_{k=0}^{2} \frac{e^{-4}4^k}{k!} = 1 - 0.238 = 0.762$.

6.4.20 $\beta = P(\text{accept } H_0 \mid H_1 \text{ is true}) = P(Y < \ln 10 \mid \lambda) = \int_0^{\ln 10} \lambda e^{-\lambda y} dy = 1 - e^{-\lambda \ln 10} = 1 - 10^{-\lambda}$.

6.4.21 $\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y_1 + Y_2 \le k \mid \theta = 2)$. When H_0 is true, Y_1 and Y_2 are uniformly distributed over the square defined by $0 \le Y_1 \le 2$ and $0 \le Y_2 \le 2$, so the joint pdf of Y_1 and Y_2 is a plane parallel to the Y_1Y_2-axis at height $\frac{1}{4}\left(= f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = \frac{1}{2} \cdot \frac{1}{2}\right)$. By geometry, α is the volume of the triangular wedge in the lower left-hand corner of the square over which Y_1 and Y_2 are defined. The hypotenuse of the triangle in the Y_1Y_2-plane has the equation $y_1 + y_2 = k$. Therefore, α = area of triangle × height of wedge = $\frac{1}{2} \cdot k \cdot k \cdot \frac{1}{4} = k^2/8$. For α to be 0.05, $k = \sqrt{0.04} = 0.63$.

6.4.22 $\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(Y_1Y_2 \le k^* \mid \theta = 2)$. If $\theta = 2$, the joint pdf of Y_1 and Y_2 is the horizontal plane $f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{4}$, $0 \le y_1 \le 2$, $0 \le y_2 \le 2$. Therefore, $\alpha = P(Y_1Y_2 \le k^* \mid \theta = 2) =$
$2 \cdot \frac{k^*}{2} \cdot \frac{1}{4} + \int_{k^*/2}^{2} \int_0^{k^*/y_1} \frac{1}{4} dy_2 dy_1 = \frac{k^*}{4} + \int_{k^*/2}^{2} \frac{k^*}{4y_1} dy_1 = \frac{k^*}{4} + \left(\frac{k^*}{4} \ln y_1 \Big|_{k^*/2}^{2}\right) =$
$\frac{k^*}{4} + \frac{k^*}{4}\ln 2 - \frac{k^*}{4}\ln\frac{k^*}{2}$. By trial and error, $k^* = 0.087$ makes $\alpha = 0.05$.

Note: The k* value in 6.4.22 is incorrect. The correct value is approximately 0.0349.