ST 430
Homework #7 Solutions

6.5.2 Let $y = \sum_{i=1}^{10} y_i$. Then $L(\hat{\omega}) = \prod_{i=1}^{10} \lambda_o e^{-\lambda_o y_i} = \lambda_0^{10} e^{-\lambda_0 \sum_{i=1}^{10} y_i} = \lambda_0^{10} e^{-\lambda_0 y}$. Also, $L(\lambda) = \prod_{i=1}^{10} \lambda e^{-\lambda y_i} = \lambda^{10} e^{-\lambda y}$, so $\ln L(\lambda) = 10 \ln \lambda - \lambda y$ and $\frac{d \ln L(\lambda)}{d\lambda} = \frac{10}{\lambda} - y$. Setting the latter equal to 0 implies that the maximum likelihood estimate for λ is $\lambda_e = \frac{10}{y}$. Therefore,

$L(\hat{\Omega}) = \left(\frac{10}{y}\right)^{10} e^{-\left(\frac{10}{y}\right)y} = (10/y)^{10}e^{-10}$. The generalized likelihood ratio, then, is the quotient $\lambda_0^{10} e^{-\lambda_0 y} / (10/y)^{10} e^{-10} = (\lambda_0 e/10)^{10} y^{10} e^{-\lambda_0 y}$. It follows that H_0 should be rejected if $\lambda = y^{10} e^{-\lambda_0 y} \le \lambda^*$, where λ^* is chosen so that $\int_0^{\lambda^*} f_\lambda(\lambda \mid H_0 \text{ is true}) d\lambda = 0.05$.

6.5.3 $L(\hat{\omega}) = \prod_{i=1}^{n} (1/\sqrt{2\pi}) e^{-\frac{1}{2}(y_i - \mu_0)^2} = (2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu_0)^2}$. Since $\bar{y}$ is the maximum likelihood estimate for μ (recall the first derivative taken in Example 5.2.4),

$L(\hat{\Omega}) = (2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\bar{y})^2}$. Here the generalized likelihood ratio reduces to

$\lambda = L(\hat{\omega})/L(\hat{\Omega}) = e^{-\frac{1}{2}((\bar{y}-\mu_0)/(1/\sqrt{n}))^2}$. The null hypothesis should be rejected if $e^{-\frac{1}{2}((\bar{y}-\mu_0)/(1/\sqrt{n}))^2} \le \lambda^*$ or, equivalently, if $\left|(\bar{y} - \mu_0)\right|/(1/\sqrt{n}) > \lambda^{**}$, where values for λ^{**} come from the standard normal pdf, $f_Z(z)$.

6.5.4 To test H_0: $\mu = \mu_0$ versus H_1: $\mu = \mu_1$, the "best" critical region would consist of all those samples for which $\prod_{i=1}^{n}(1/\sqrt{2\pi})^n e^{-\frac{1}{2}(y_i-\mu_0)^2} / \prod_{i=1}^{n}(1/\sqrt{2\pi})^n e^{-\frac{1}{2}(y_i-\mu_1)^2} \le k$. Equivalently, H_0 should be rejected if $\sum_{i=1}^{n}(y_i - \mu_0)^2 - \sum_{i=1}^{n}(y_i - \mu_1)^2 > 2\ln k$. Simplified, the latter becomes

$2(\mu_1 - \mu_0)\sum_{i=1}^{n} y_i > 2\ln k + n\left(\mu_1^2 - \mu_0^2\right)$. Consider the case where $\mu_1 < \mu_0$. Then $\mu_1 - \mu_0 < 0$, and the decision rule reduces to rejecting H_0 when $\bar{y} < \frac{2\ln k + n\left(\mu_1^2 - \mu_0^2\right)}{2n(\mu_1 - \mu_0)}$.

7.3.2 Subsituting $\frac{n}{2}$ and $\frac{1}{2}$ for r and λ, respectively, in the moment-generating function for a gamma pdf gives $M_{\chi_n^2}(t) = (1 - 2t)^{-n/2}$. Also, $M_{\chi_n^2}^{(1)}(t) = (-n/2)(1 - 2t)^{-n/2 - 1}(-2) =$ $n(1 - 2t)^{-n/2 - 1}$ and $M_{\chi_n^2}^{(2)}(t) = \left(-\frac{n}{2}-1\right)(n)(1-2t)^{-n/2 - 2}(-2) = (n^2 + 2n) \cdot (1 - 2t)^{-n/2 - 2}$, so $M_{\chi_n^2}^{(1)}(0) = n$ and $M_{\chi_n^2}^{(2)}(0) = n^2 + 2n$. Therefore, $E(\chi_n^2) = n$ and $\text{Var}\left(\chi_n^2\right) =$ $n^2 + 2n - n^2 = 2n$.

7.3.4 Let $Y = \frac{(n-1)S^2}{\sigma^2}$. Then $\text{Var}(Y) = \text{Var}\left(\chi_{n-1}^2\right) = 2(n - 1) = \frac{(n-1)^2\text{Var}(S^2)}{\sigma^4}$. If follows that $\text{Var}(S^2) = \frac{2\sigma^4}{n-1}$.

7.3.8 $P\left(2.51 < \frac{V/7}{U/9} < 3.29\right) = P(2.51 < F_{7,9} < 3.29) = P(F_{7,9} < 3.29) - P(F_{7,9} \le 2.51) = 0.95 - 0.90 =$ 0.05. But $P(3.29 < F_{7,9} < 4.20) = 0.975 - 0.95 = 0.025$.

7.3.11 $F = \frac{V/m}{U/n}$, where U and V are independent χ^2 random variables with m and n degrees of freedom, respectively. Then $\frac{1}{F} = \frac{U/n}{V/m}$, which implies that $\frac{1}{F}$ has an F distribution with n and m degrees of freedom.

7.3.12 If $P(a \le F_{m,n} \le b) = q$, then $P\left(a \le \frac{1}{F_{n,m}} \le b\right) = q = P\left(\frac{1}{b} \le F_{n,m} \le \frac{1}{a}\right)$. From Appendix Table A.4, $P(0.052 \le F_{2,8} \le 4.46) = 0.95$. Also, $P(0.234 \le F_{8,2} \le 19.4) = 0.95$. But $\frac{1}{4.46} = 0.224$ and $\frac{1}{0.052} = 19.23 \doteq 19.4$.