ST640: Homework 1
Due: January 27, beginning of class

1. Let \( A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \).
   
   (a) Show that \( A \) is positive definite.
   
   (b) Determine the eigenvalues and eigenvectors of \( A \).
   
   (c) Let \( p_i \) be the \( i \)th eigenvector of \( A \), \( i = 1, 2 \). Show that \( p_i p_i^T \) is a symmetric, idempotent matrix with rank 1, \( i = 1, 2 \).
   
   (d) Find \( A \otimes B \) where \( B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \).

2. Prove: the eigenvalues of a positive semi-definite matrix are nonnegative.

3. Prove: if \( A \) is positive semi-definite then \( \text{tr}(A) \geq 0 \).

4. Suppose \( A_{m \times n} = \begin{pmatrix} B_{r \times r} & C \\ D & E \end{pmatrix} \) with \( \text{rank}(A) = \text{rank}(B) = r \). Then prove that the matrix \( G_{n \times m} = \begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix} \) is a generalized inverse of \( A \). Hint: you might want to consider the nonsingular \( m \times m \) matrix \( H = \begin{pmatrix} B^{-1} & 0 \\ DB^{-1} & -I_{m-r} \end{pmatrix} \).

5. For arbitrary random vectors, prove that \( \text{Cov}(x, y) = E xy^T - E x E y^T \).

6. Let \( \underline{x} \) be a random \( n \)-vector, and let \( y_i = x_1, y_i = x_i - x_{i-1} \) (\( i = 2, \ldots, n \)). If \( \text{Var}(y) = I \) and the \( y_i \) are mutually independent, find \( \text{Var}(\underline{x}) \).

7. Verify the first two moments of \( \chi^2_{n, \lambda} \) using the moment generating function.