ST640: Homework 2
Due: February 8, beginning of class

Read Chapter 2.

1. Suppose $y \sim N_n(\mathbf{0}, \mathbf{I})$, $\mathbf{B}$ is a symmetric matrix, and $y^T \mathbf{B} y \sim \chi^2_p$. Prove that $\mathbf{B}$ is an idempotent matrix of rank $p$.

Do this proof “from scratch.” In other words, you can’t use any of the theorems in Hocking or your class notes that relate quadratic forms and idempotent matrices (e.g., Theorem 2.4 in Hocking).

2. Let $z$ be a random $n$-vector and let $\mathbf{A}_{n \times n}$ be symmetric. Prove that if $E(z) = \theta$ and $\text{Var}(z) = \Sigma$, then $E\left[ z^T \mathbf{A} z \right] = \text{tr}(\mathbf{A} \Sigma) + \theta^T \mathbf{A} \theta$.

Hint: It might help to consider $E\left[ (z - \theta)^T \mathbf{A} (z - \theta) \right]$.

3. Suppose $y \sim N_n(\theta, \Sigma)$ and consider the partitions $y = (z_1 \quad z_2)^T$, $\theta = (\alpha_1 \quad \alpha_2)^T$, and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ where $z_1$ and $\alpha_1$ are $p$-vectors and $\Sigma_{11}$ is $p \times p$ with $p < n$.

If $w = z_1 - \Sigma_{12} \Sigma_{22}^{-1} z_2$, prove that $\text{Cov}(w, z_2) = 0$ and hence deduce that $w$ and $z_2$ are independent. Hence prove that the conditional distribution of $z_1 = w + \Sigma_{12} \Sigma_{22}^{-1} z_2$ given $z_2 = z_2^*$ is $N_p \left( \alpha_1 + \Sigma_{12} \Sigma_{22}^{-1} (z_2^* - \alpha_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \right)$.

4. Problem 2.8, parts b and c, in Hocking.