Testing for spatial asymptotic independence of extremes using the madogram function

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Motivation

Extremes in environmental and climate fields

Spatiotemporal models

with special attention to

spatial extremal behaviour

Aim :

Test for asymptotic independence from spatial data

Extremal coefficient

(X, Y) Bivariate Extreme Value distribution with same margin F

$$\chi = \lim_{u \to u^*} P(X > u | Y > u) = \lim_{u \to u^*} \frac{P(X > u, Y > u)}{P(Y > u)}$$

Asymptotic independence $\Leftrightarrow \chi = 0$

If F Fréchet unity, $F(x) \equiv GEV_{1,1,1}(x) = \exp(-1/x)$ (no loss of generality)

$$P(X < x, Y < x) = F(x)^{\theta}$$

$$\boxed{\chi = 2 - \theta}$$

$\theta = 2$: Asymptotic Independence
$1 \leq \theta < 2$: non Asymptotic Independence

Madogram test

(X, Y) vector with bivariate extreme distribution G, margins F_X and F_Y.

$$Z = \frac{1}{2}|F_X(X) - F_Y(Y)| \equiv \frac{1}{2}|U - V|$$

$$E(Z) = \frac{\theta - 1}{2(\theta + 1)}$$

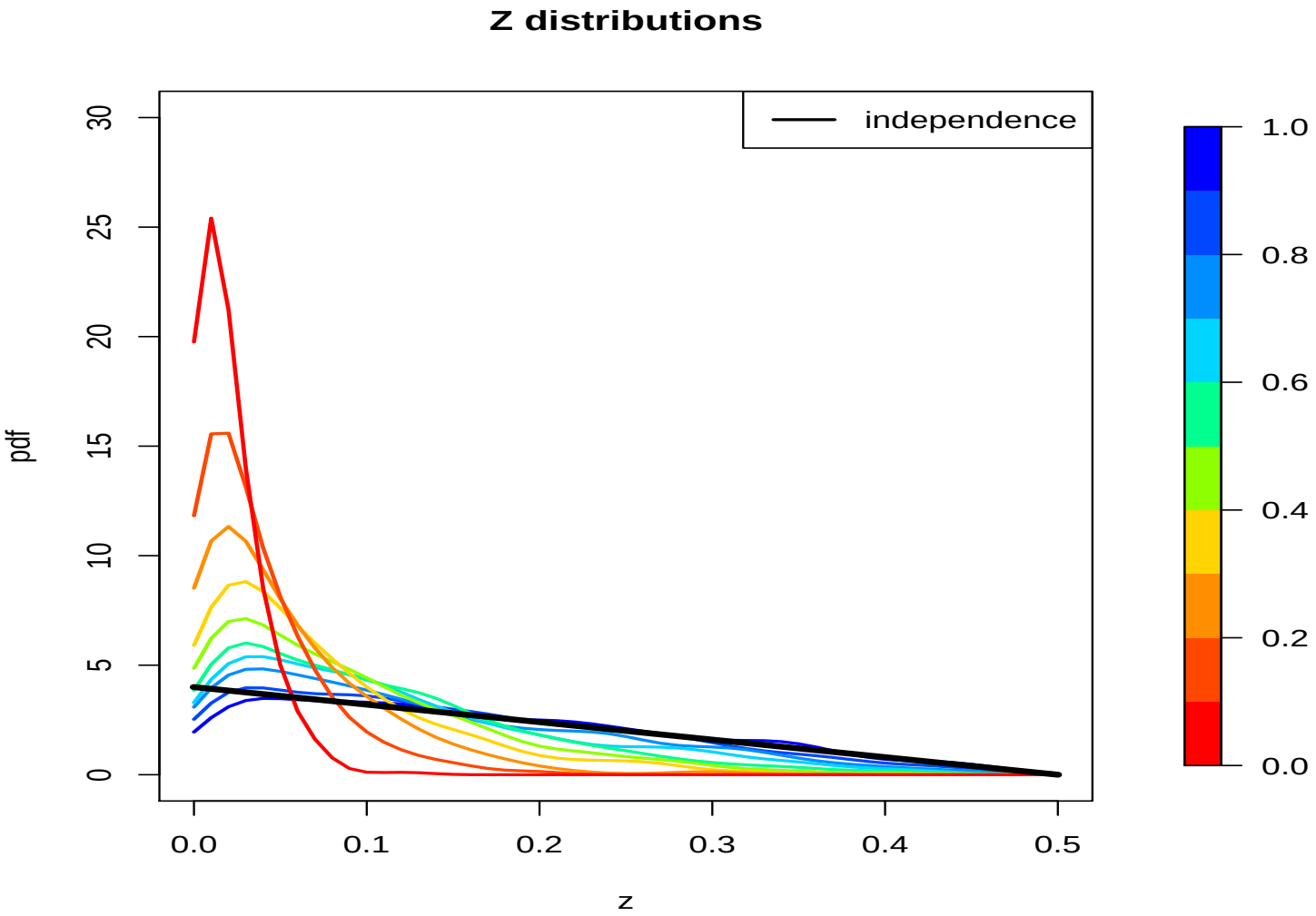
$$V(Z) = \frac{1}{6} - \left[\frac{\theta - 1}{2(\theta + 1)}\right]^2 - \int_0^1 \frac{1}{(1 + A(1 - t))^2} dt$$

If asymptotic independence

- $E(Z) = \frac{1}{6}$
- $V(Z) = \frac{1}{72}$
- $f_Z(z) = 4 - 8z$, $z \in [0, \frac{1}{2}]$

Madogram test

Bivariate logistic distribution



Madogram test

Asymptotic independence test : madogram $\nu = E(Z)$

$$H_0 : \nu = \frac{1}{6} \qquad H_1 : \nu < \frac{1}{6}$$

$(X_1, Y_1), \ldots, (X_n, Y_n)$ i.i.d.

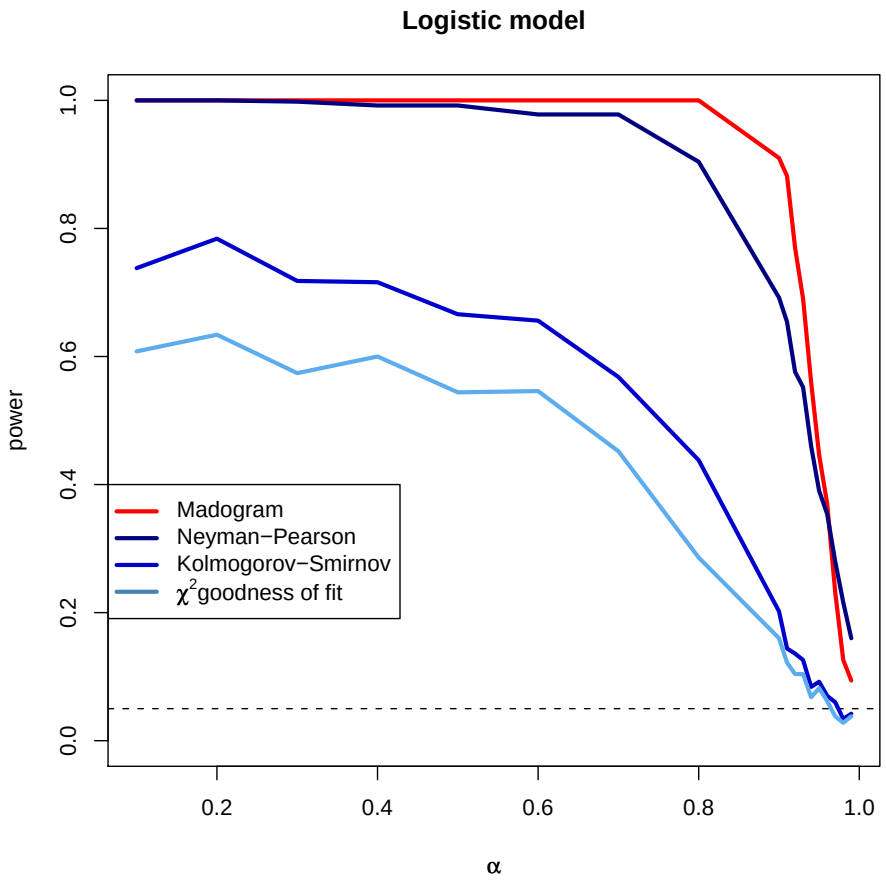
ν estimated by

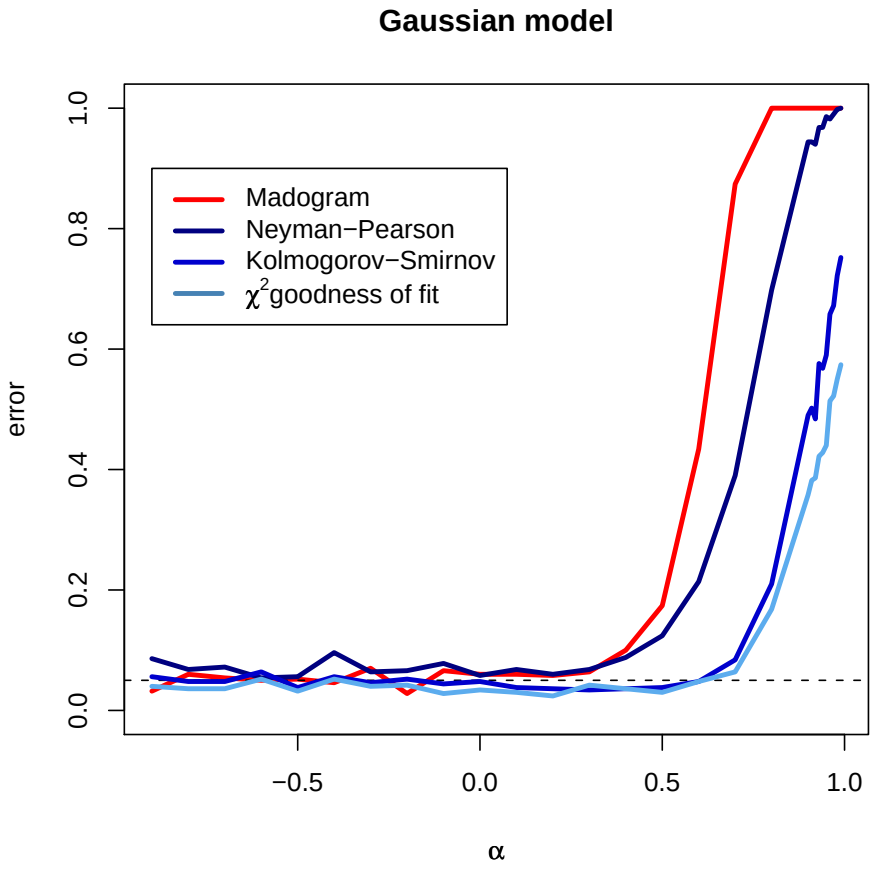
$$\widehat{\nu} = \frac{1}{2n} \sum_{i=1}^{n} |\widehat{F_X}(X_i) - \widehat{F_Y}(Y_i)|$$

Central limit theorem

$$\sqrt{n} \frac{\widehat{\nu} - 1/6}{\widehat{\sigma_{\widehat{\nu}}}} \to \mathcal{N}(0, 1)$$

Madogram test





Spatial case : multiple realisations

X a stationary maxima random field, with Fréchet margins $X(s_1), X(s_2), \ldots, X(s_n)$ observations.

Case of multiple independent realisations $\left(X^\ell(s_i)\right)_{\ell=1,L}$

- for each pair of points (s_j, s_k) estimate $\widehat{\nu_F}(X(s_j), X(s_k))$
- test : $H_0 : \nu_{jk} = 1/6 \qquad H_1 : \nu_{jk} < 1/6$
- derive p-values p_{jk}

Multiple hypothesis testing procedure (FDR, Positive Regression Dependency on Subset) :

$\hookrightarrow$ give pairs of points significatively non asymptotically independent.

Spatial case : multiple realisations

Asymptotic independence at distance h or for a class of distances $\mathcal{C}_i =]h_i; h_{i+1}]$?

R : number of rejected hypothesis under the null

Resampling procedure

1. fix a site s_j and resample the vector $\mathbf{X} = {}^t(X^\ell(s_j)), \ell = 1, \ldots, n$
2. compute the madogram with all other sites in $\mathcal{C}_i$
3. compute the associated p-values
4. repeat (1) to (3) for pair of points in the class $\mathcal{C}_i$
5. consider the obtained sample of p-values and use the FDR approach to count the rejected hypotheses R^*
6. repeat (1) to (5) a number k times.

Empirical law of R^* number of rejected hypothesis under null.
Conclude with

$$P(R^* \geq R)$$

Simulations : models

1. Gaussian field with autocorrelation function $\rho(\cdot)$

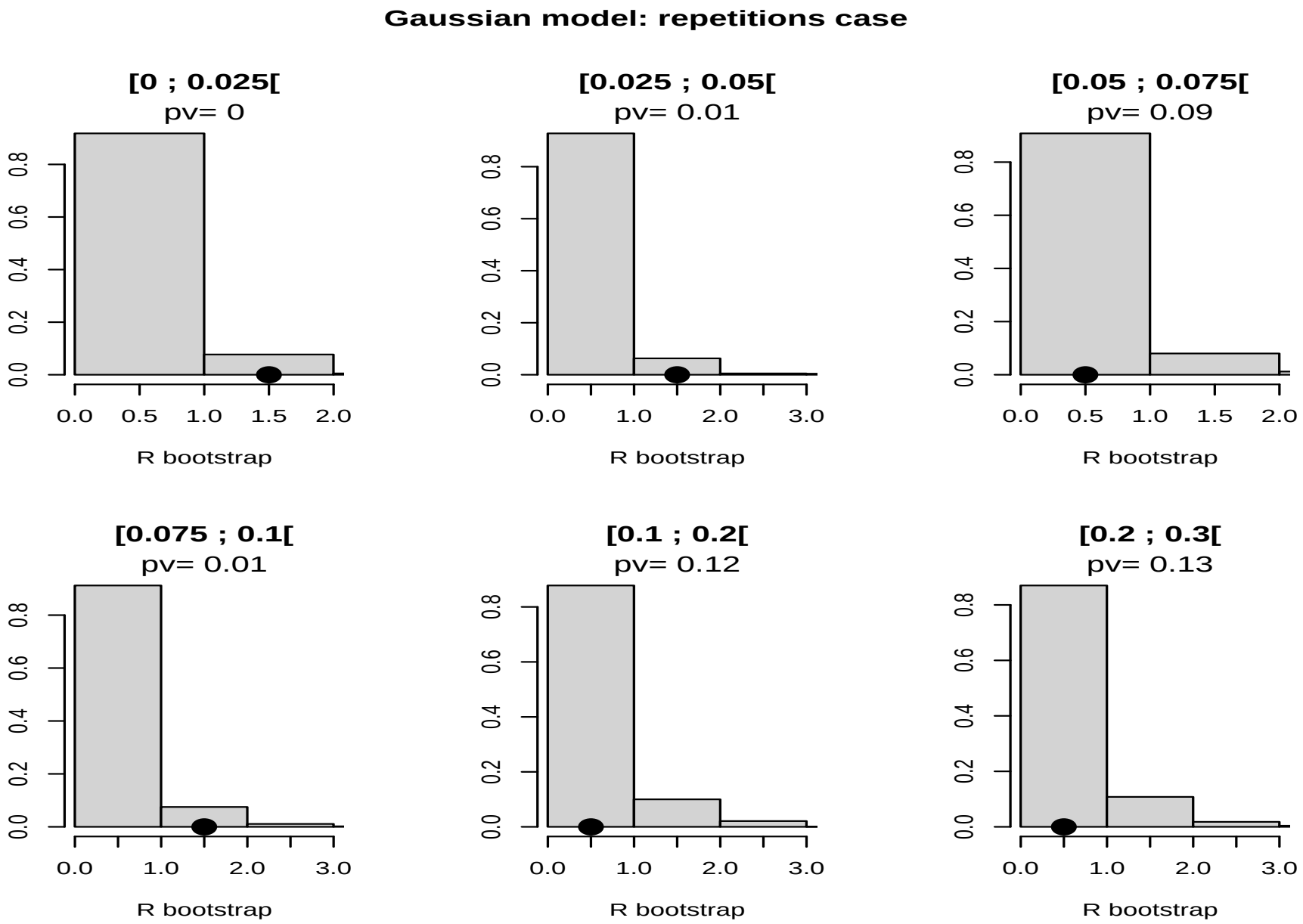
$$\theta(h) \equiv 2$$

2. Storm process (Smith (1991), Schlather (2002))

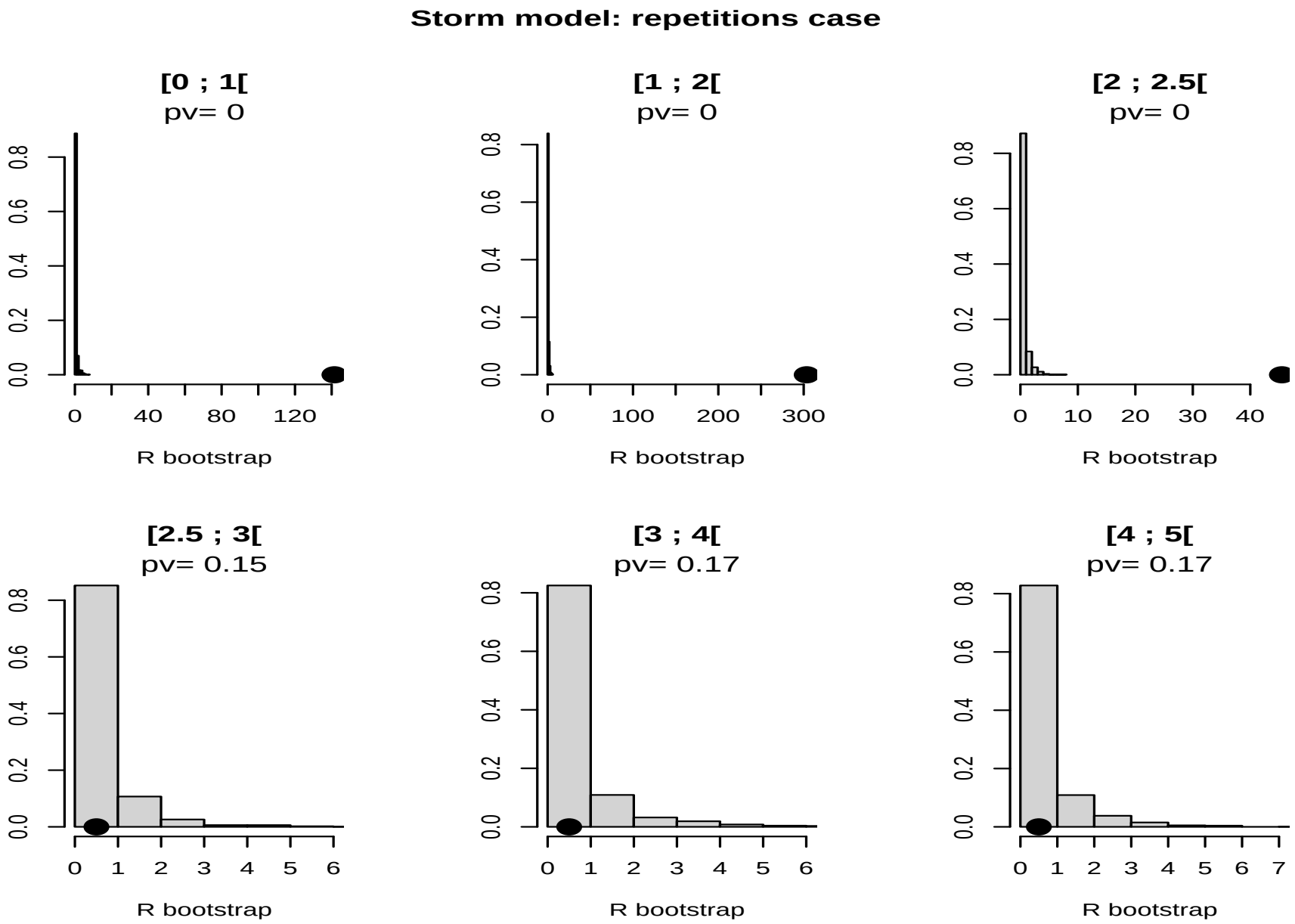
$$Z(s) = \sup_{j=1,2,\ldots} \zeta_j g(x_j - s), \quad (\zeta_j, x_j)_j \text{ Poisson on } \mathbb{R}^{+*} \times \mathbb{R}^2$$

$$\theta(h) = 2\Phi(\sqrt{{}^t h M^{-1} h}/2)$$

Simulations : gaussian process, multiple realisations



Simulations : Storm process, multiple realisations



Spatial case : single maxima realisation

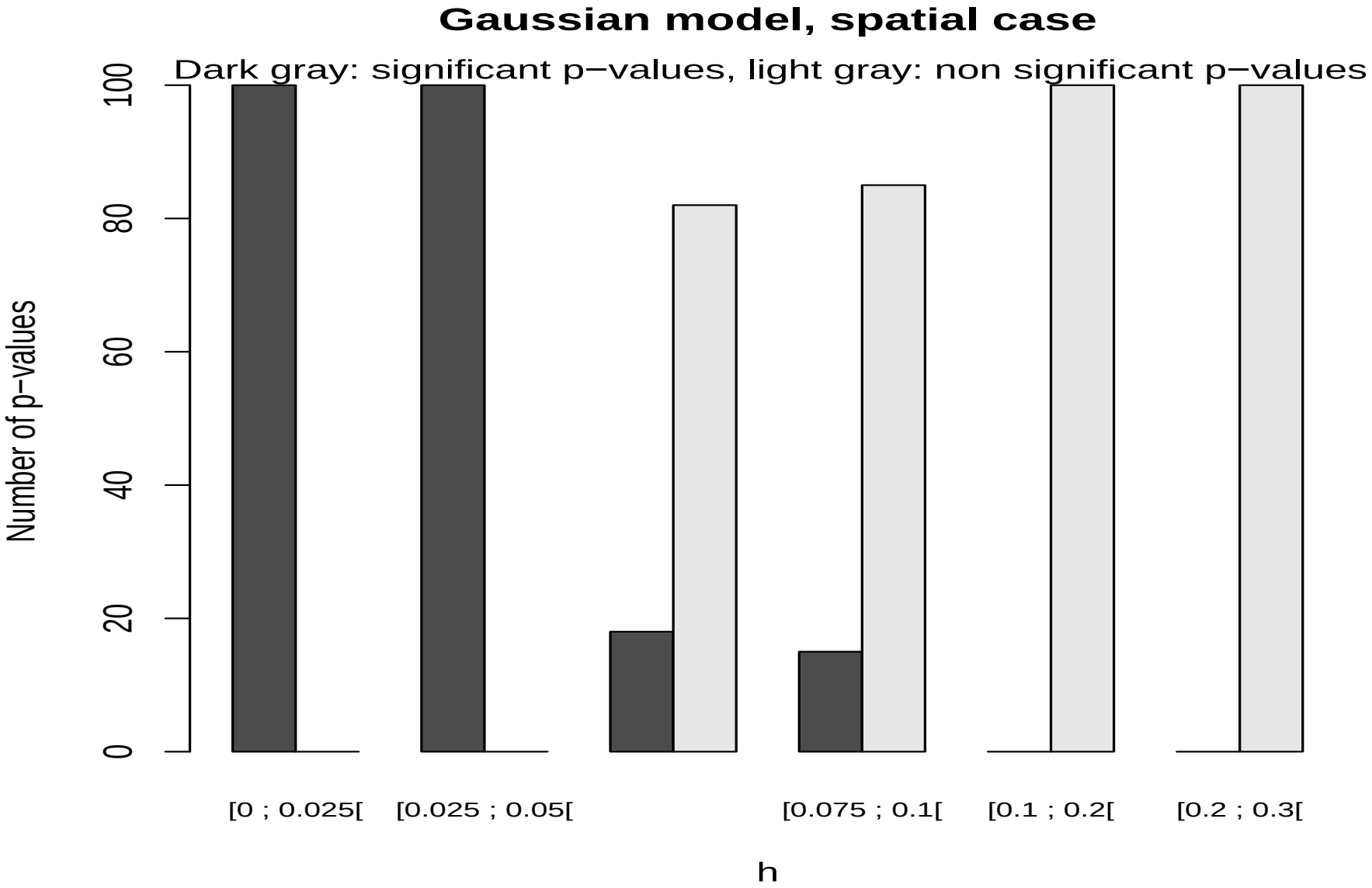
Case of a single realisation of maxima

- compute the madogram function on classes of distances $\mathcal{C}_i =]h_i; h_{i+1}]$

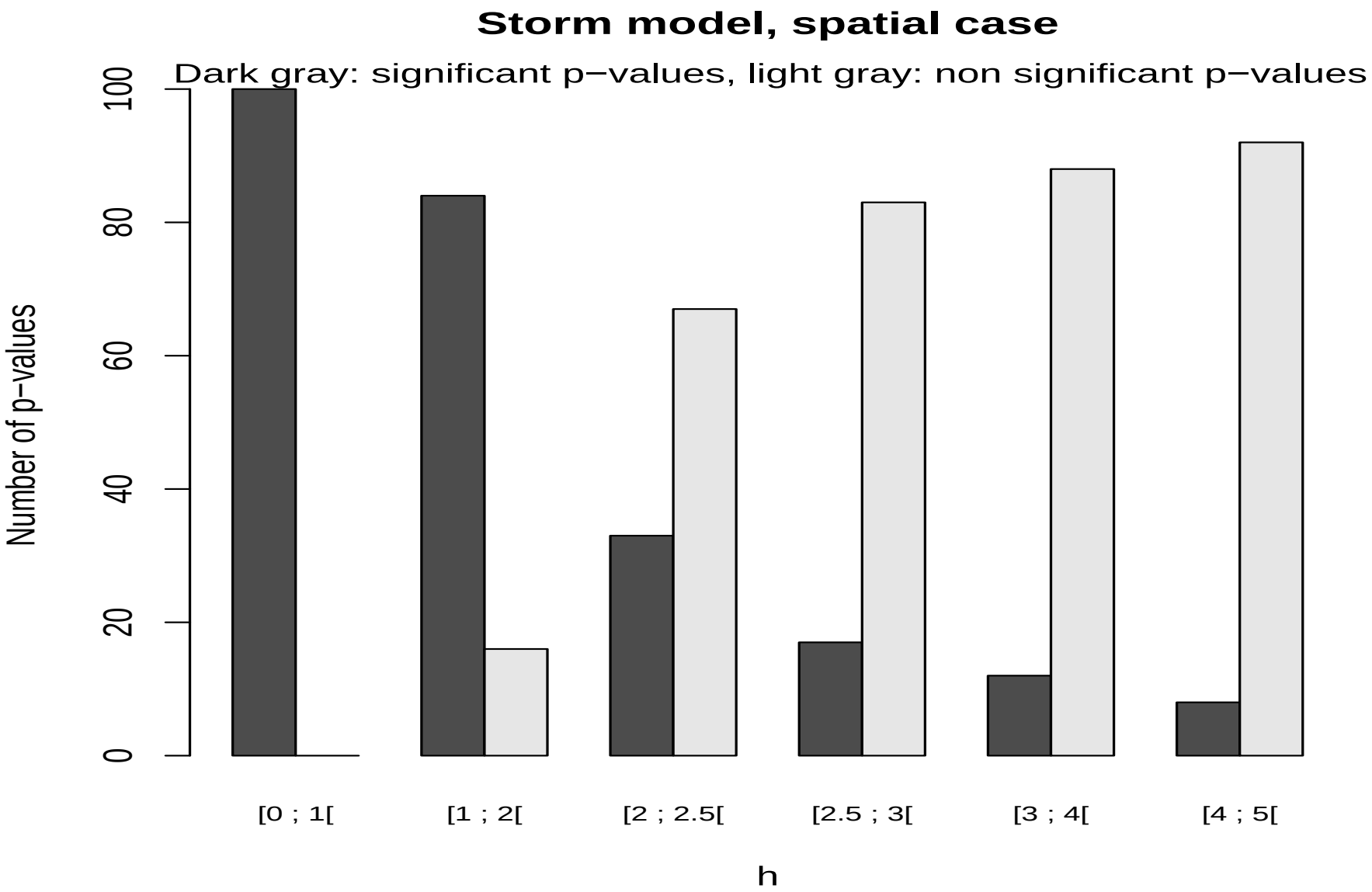
$$\widehat{\nu}(h) = \frac{1}{2n_i} \sum_{|s_j - s_k| \in \mathcal{C}_i}^{n} |\widehat{F}(X(s_j)) - \widehat{F}(X(s_k))|$$

- compute the associated p-values
- FDR multiple-test procedure to determine which are significant

Simulations : gaussian process, single realisation



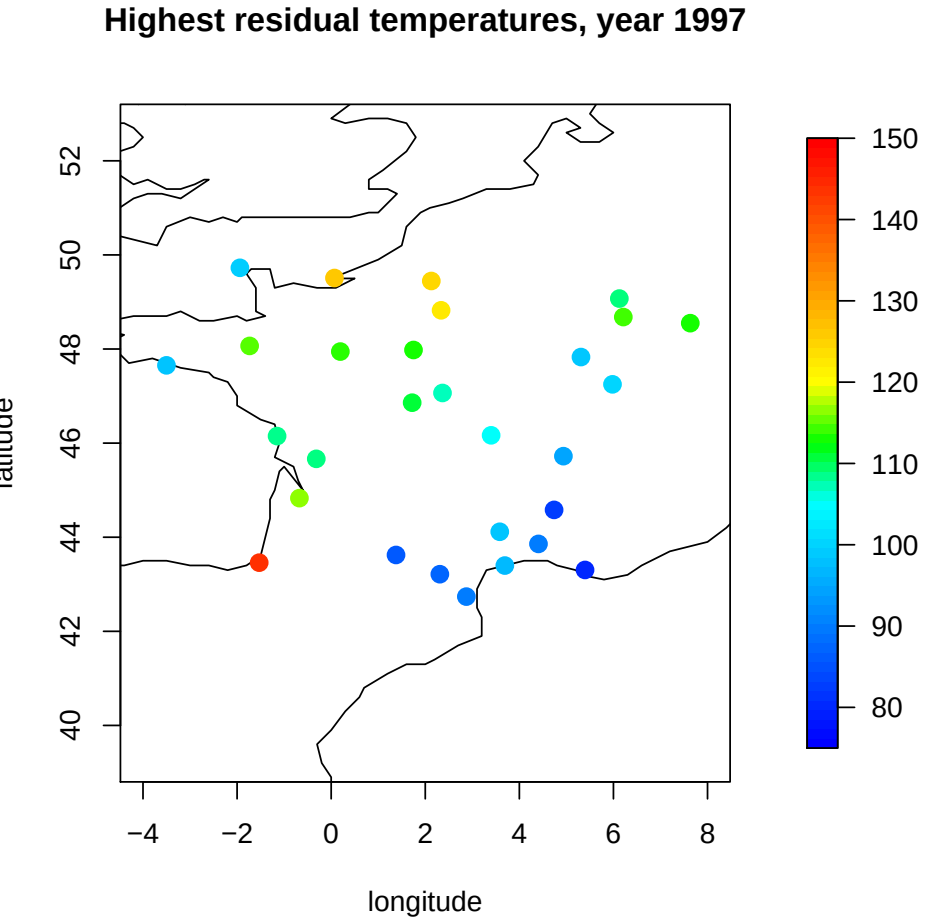
Simulations : storm process, single realisation

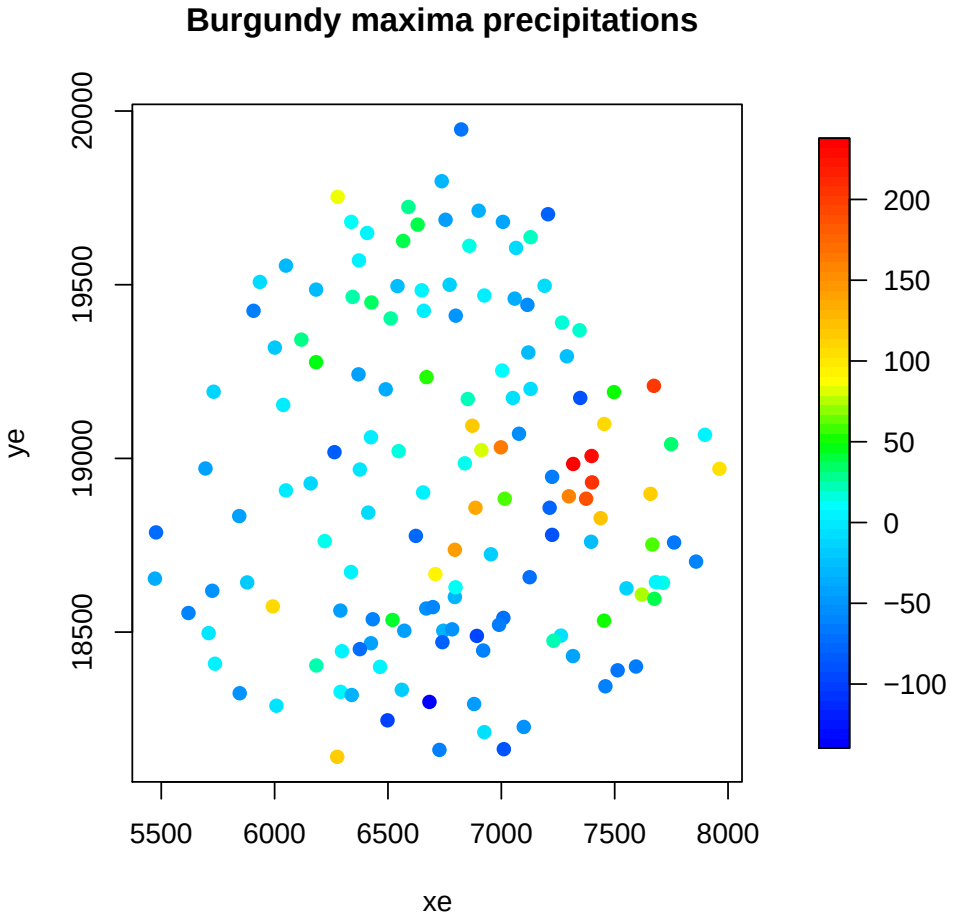


Examples

maxima daily temperatures in 29 French towns for 100 years

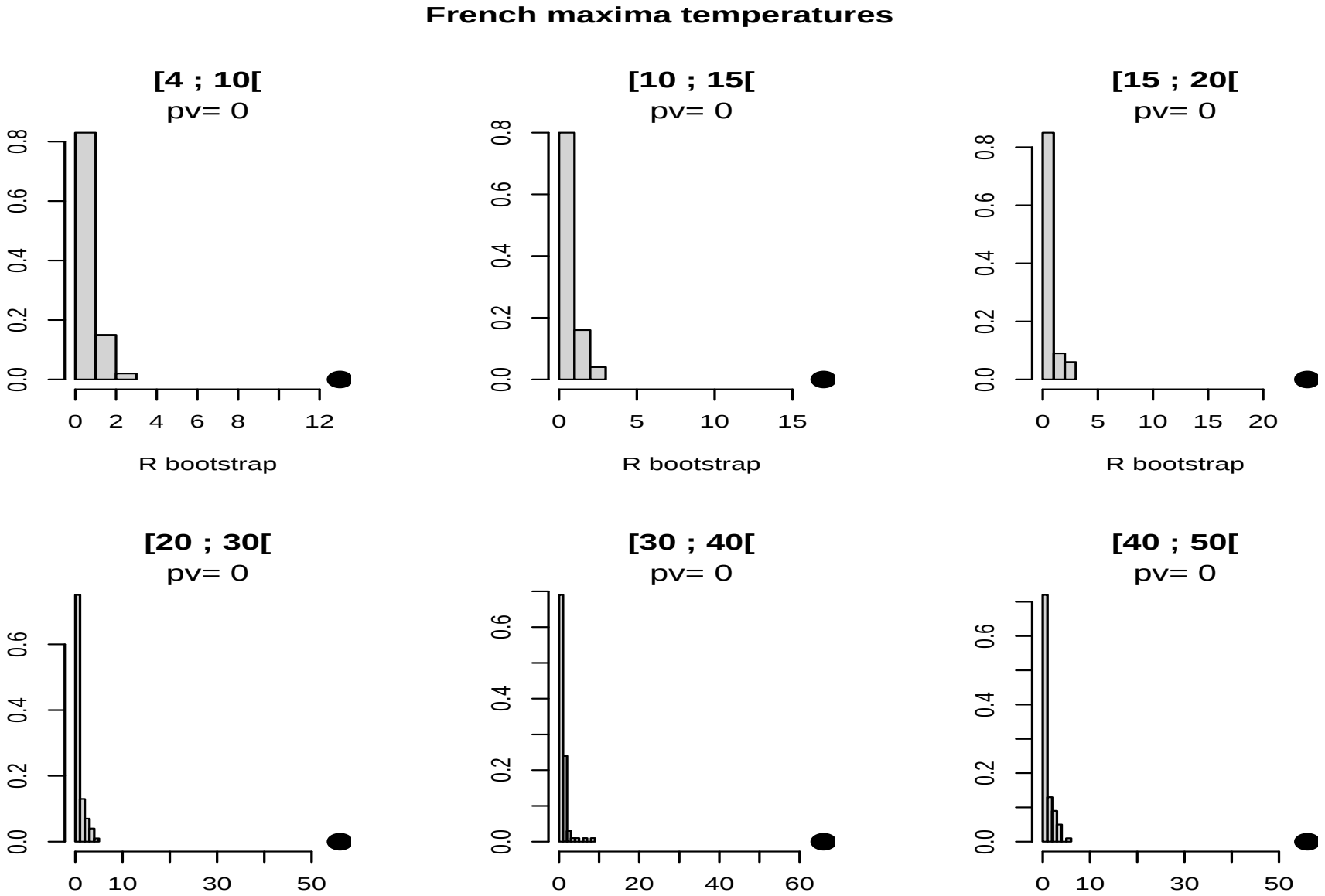
Burgundy maxima precipitations for 30 years





Example : French maxima temperatures

Repetitions case



Example : Burgundy maxima precipitations

Single maxima realisation