Aggregation of Rapidly varying Risks and Asymptotic independence

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(Joint work with S. I. Resnick)

Outline:

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- Consider a portfolio with two assets.
  The risks associated with the two assets: X and Y.
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  - $P(X + Y > x)$ for large x.
- Direct computation is difficult in most cases.
- Simulation strategies are not well-known in the case where the marginal distributions are subexponential (defined later).

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- Start with the simplifying assumption : X and Y are identically distributed.
- Approach the problem using extreme value theory

Multivariate Extremes

- Suppose $(X, Y) \sim F(x, y)$.

Definition

F is in the *maximal domain of attraction* of extreme value distribution G, if there exists normalizing constants $a_n^{(i)} > 0$, $b_n^{(i)} \in \mathbb{R}, 1 \leq i \leq 2$, such that as $n \to \infty$,

$$F^n(a_n^{(1)} x^{(1)} + b_n^{(1)}, a_n^{(2)} x^{(2)} + b_n^{(2)}) \to G(x^{(1)}, x^{(2)})$$

for limit distribution G, such that each marginal G_i, i = 1, 2 is non-degenerate.

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- Implications of the definition : If the marginals of F are F_i, i = 1, 2,

$$F_i^n(a_n^{(i)} x^{(i)} + b_n^{(i)}) \to G_i(x^{(i)})$$

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- The last case (Weibull) $\Rightarrow F_1$ has a finite right end point, so the limit makes little sense here. Hence, the case where

$$G_1(x) = \begin{cases} \exp(-(-x)^{\alpha}) & x < 0, \\ 0 & x \geq 0 \end{cases}$$

for some $\alpha > 0$, is less interesting for the present problem.

Asymptotic independence

- Suppose, F is in the maximal domain of attraction of extreme value distribution G, i.e. there exists normalizing constants $a_n^{(i)} > 0$, $b_n^{(i)} \in \mathbb{R}, 1 \leq i \leq 2$, such that as $n \to \infty$,

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- In general, *asymptotic independence* is present in many models used in practice.

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- In the *Gumbel* case, i.e. $G_1(x) = \exp(-e^{-x}), x \in \mathbb{R}$, in *absence* of *asymptotic independence*, the question is recently answered by Klüppelberg and Resnick (2008).

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- Remaining case: When $X \stackrel{d}{=} Y \in MDA(\Lambda)$, where $\Lambda(\cdot)$ is the Gumbel distribution and X and Y are *asymptotically independent* ???
  Denote by $\mathcal{C}$ the class of 2 dimensional distributions of (X,Y) for which $X \stackrel{d}{=} Y, (X,Y) \in MDA(G)$ where $G(x,y) = \Lambda(x)\Lambda(y)$.

One special case: Independent X and Y

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Definition

A distribution F on $\mathbb{R}_+$ is *subexponential if*

$$\lim_{x \to \infty} \frac{1 - F^{*2}(x)}{1 - F(x)} = 2.$$

  - Let us denote the class of subexponential distributions by $\mathcal{S}$.

Back to the general case: Problems

- There are 2 distinct behaviors observed within the class $\mathcal{C}$ $(X \stackrel{d}{=} Y, (X, Y) \in MDA(G)$ where $G(x, y) = \Lambda(x)\Lambda(y))$:

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  - Very different tail behavior is observed by Albrecher, Asmussen and Kortschak (2006), who exhibit a distribution of (X,Y), with X and Y being identically distributed and asymptotically independent with common distribution $F \in MDA(\Lambda) \cap \mathcal{S}$, but
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  $$\lim_{x\to\infty} \frac{P(X+Y>x)}{P(X>x)} = \infty.$$
- So, subexponentiality of X and Y is *not* sufficient to ensure a limit in $(0,\infty)$ when (X,Y) has a distribution belonging to $\mathcal{C}$.

What assumptions would ensure the limit = 2?

- We provide sufficient conditions under which the limit is 2.

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- We provide sufficient conditions under which the limit is 2.
- To state the conditions, first define the auxiliary function f of F. For a distribution $F \in MDA(\Lambda)$, there exists an absolutely continuous function f with its derivative going to 0, such that

$$\lim_{t \to \infty} \frac{\bar{F}(t + xf(t))}{\bar{F}(t)} = e^{-x}. \tag{1}$$

Such a function f is called the auxiliary function of F (de Haan (1970)). In common cases f is the reciprocal of the hazard function

$$f = \frac{1 - F}{F'}.$$

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Theorem

Suppose, $X \stackrel{d}{=} Y \sim F \in MDA(\Lambda)$ *and* $x_0 = \sup\{x : F(x) < 1\} = \infty$. *If* (2), (3) *and* (4) *hold, then*
$$\lim_{x\to\infty} \frac{P(X + Y > x)}{P(X > x)} = 2.$$

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  - First model : $(X_1, X_2) \sim \mathcal{N}(\mu, \Sigma)$, where $\rho = Correlation(X_1, X_2) < 1$. Then, $(X, Y) = (\exp(X_1), \exp(X_2))$ satisfies all our assumptions.

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  - Second Model : Suppose, $X_i \overset{iid}{\sim} F$, $i = 1, 2, 3$, where for some $\alpha > 1$,

$$\bar{F}(x) = \begin{cases} \exp\{-(\log x)^{\alpha}\}, & \text{if } x > 1, \\ 1, & \text{if } x \leq 1. \end{cases}$$

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  - Third Model : Suppose, $F \in MDA(\Lambda)$, concentrated on $[0, \infty)$, such that $x_0 = \sup\{x : F(x) < 1\} = \infty$, $x_1 = \inf\{x : F(x) > 0\} = 0$, $\liminf_{x \to \infty} f(x) > 0$.
    Recall, $f(x)$ is the auxiliary function as defined in (1).
    Examples of such F: Exponential, Gamma, Lognormal etc.
    Then, $(X, Y) = (F^{\leftarrow}(U), F^{\leftarrow}(1 - U))$, where $U \sim \text{Uniform}(0, 1)$, satisfies all our assumptions.

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    Then, $(X, Y) = (F^{\leftarrow}(U), F^{\leftarrow}(1 - U))$, where $U \sim \text{Uniform}(0, 1)$, satisfies all our assumptions.
- The third model shows that subexponentiality of X and Y is *not* a necessary condition to ensure a limit in $(0, \infty)$.

Summary

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- The assumption (2) implies *asymptotic independence*, hence the class of distributions satisfying the assumptions of the theorem forms a subclass of the class of distributions $\mathcal{C}$ we considered first ($X \stackrel{d}{=} Y, (X,Y) \in MDA(G)$ where $G(x,y) = \Lambda(x)\Lambda(y)$).

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- The assumption $X \stackrel{d}{=} Y \in MDA(\Lambda)$ implies that both X and Y are rapidly varying, i.e. for all $t > 1$,

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- The conditions are only sufficient. Necessary and sufficient conditions?
- The result is extended to the case where X and Y are not identically distributed, i.e. where $X \sim F, Y \sim G$ and $\lim_{x\to\infty} \frac{\bar{G}(x)}{\bar{F}(x)} = c \in [0,\infty)$.

Thank You !