Homework #3
due Wednesday, October 12th
STAT 740: CONSTRAINED ESTIMATION AND INFERENCE
FALL 2011

1. Let $\mathcal{X}$ be a finite set of points on which a partial ordering $\preceq$ is defined. Let $f : \mathcal{X} \to \mathbb{R}$ be any function, let $w(x) \geq 0$ be a weight function. Prove that $\hat{f} : \mathcal{X} \to \mathbb{R}$ minimizes $\sum_{x\in\mathcal{X}}[f(x) - g(x)]^2 w(x)$ over functions $g : \mathcal{X} \to \mathbb{R}$ that are isotonic with respect to $\preceq$ if and only if
$$\sum_{x\in\mathcal{X}}[f(x) - \hat{f}(x)]\hat{f}(x)w(x) = 0.$$
and
$$\sum_{x\in\mathcal{X}}[f(x) - \hat{f}(x)]g(x)w(x) \leq 0, \text{ for all } g \text{ isotonic on } \mathcal{X} \text{ with respect to } \preceq .$$

2. Consider these points in $\mathbb{R}^2$:
$$\mathcal{X} = \{(1,1),(2,1),(3,1),(4,1),(1,2),(2,2),(3,2),(4,2)\},$$
and the ordering $(x_1, x_2) \preceq (y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$.

   (a) Enumerate all upper sets.

   (b) Suppose $y_i = f(\boldsymbol{p}_i) + \varepsilon_i$, $i = 1, \ldots, 8$, where the $\boldsymbol{p}_i$ are the points in $\mathcal{X}$, and f is isotonic with respect to $\preceq$. Let $\theta_i = f(x_i)$, and consider the problem of minimizing $\| \boldsymbol{y} - \boldsymbol{\theta} \|^2$ with $\boldsymbol{A\theta} \geq \boldsymbol{0}$, that provides the isotonic regression with respect to $\preceq$. Write down $\boldsymbol{A}$ and find the constraint cone edges.

3. Let $\mathcal{X}$ be a finite set of points on which a partial ordering $\preceq$ is defined. Let $f : \mathcal{X} \to \mathbb{R}$ be any function, let $w(x) \geq 0$ be a weight function. If $\hat{f}$ is the isotonic regression of f, show that for any upper set U in $\mathcal{X}$,
$$\sum_{x\in U}[f(x) - \hat{f}(x)]w(x) \leq 0.$$

4. Suppose a cone $\mathcal{C}$ in $\mathbb{R}^2$ has edges $\boldsymbol{\delta}_1 = (1,1)'$ and $\boldsymbol{\delta}_2 = (0,1)'$. The random vector $\boldsymbol{y}$ takes values in $\mathbb{R}^2$ and has mean $\boldsymbol{0}$ and identity covariance matrix. Let $\hat{\boldsymbol{y}}$ be the projection of $\boldsymbol{y}$ onto $\mathcal{C}$, and explicitly find the distribution of $\| \hat{\boldsymbol{y}} \|^2$ (including the mixing coefficients).

5. Consider the model

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i,$$

where we can assume that the ε_i are *iid* mean-zero normal random errors. We want to test the null hypothesis that $E(y)$ is constant. The usual F-test uses the alternative that $E(y)$ is unconstrained quadratic in x, but suppose that we also know that $E(y)$ must be non-decreasing in x. What is the appropriate test statistic and its distribution under H_0?

Using only the cone projection code on the class website, write code to find the mixing coefficients and perform the test. Then use your code to get a p-value for the test with the data posted on the web site. Also get the usual F-statistic p-value to compare.