Homework #2

due Wednesday, September 21th

STAT 740: CONSTRAINED ESTIMATION AND INFERENCE

FALL 2011

1. Let $\boldsymbol{y} = (10, 8, 14, 12, 13)'$, and consider

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

   Let $\mathcal{C} = \{\boldsymbol{\theta} \in \mathbb{R}^5 : \boldsymbol{A\theta} \geq \boldsymbol{0}\}$.

   (a) Find a basis for V, the null space of $\boldsymbol{A}$.

   (b) Find a set of edges of the cone $\Omega = \mathcal{C} \cap V^\perp$.

   (c) Find a set of edges for the polar cone Ω^o.

   (d) Find the unique $J \subseteq \{1, 2, 3, 4\}$ and the representation of $\boldsymbol{y}$ given in Theorem 6 of the "Constraint and Polar Cone" lecture.

   (e) What is $\hat{\boldsymbol{\theta}}$ to minimize $\| \boldsymbol{y} - \boldsymbol{\theta} \|^2$?

2. Suppose we have a data vector $\boldsymbol{y} \in \mathbb{R}^n$ and an $m \times n$ full row rank matrix $\boldsymbol{A}$, and we want to find $\hat{\boldsymbol{\theta}} \in \mathbb{R}^n$ to minimize $\| \boldsymbol{y} - \boldsymbol{\theta} \|^2$ over the set $\{\boldsymbol{\theta} : \boldsymbol{A\theta} \geq \boldsymbol{c}\}$, for some vector $\boldsymbol{c} \in \mathbb{R}^m$ where the constraint set is non-empty. How would you transform this problem so that you can solve it using the cone projection methods discussed in the class?

3. Suppose $\boldsymbol{A}$ is an $m \times n$ constraint matrix and define $\mathcal{C} = \{\boldsymbol{\theta} \in \mathbb{R}^n : \boldsymbol{A\theta} \geq \boldsymbol{0}\}$. Prove the claim in the "more constraints than dimensions" lecture: If a row of $\boldsymbol{A}$ is a positive linear combination of other rows, it can be removed without changing $\mathcal{C}$.

4. Suppose we have a data vector $\boldsymbol{y} \in \mathbb{R}^n$, and we want to find $\hat{\boldsymbol{\theta}} \in \mathbb{R}^n$ to minimize $\| \boldsymbol{y} - \boldsymbol{\theta} \|^2$ over the constraints $\boldsymbol{A}_1\boldsymbol{\theta} \geq \boldsymbol{0}$ and $\boldsymbol{A}_2\boldsymbol{\theta} = \boldsymbol{0}$, where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix}$$

   is full row rank. Transform the problem to a standard cone projection, and explain how to find $\hat{\boldsymbol{\theta}}$.

5. Consider the "warped plane" model

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{1i} x_{2i} + \varepsilon_i$$

and we want to constrain the expected response to be increasing in both predictors. Without loss of generality, we can scale the predictors to fall in $[0, 1]$, and then the constraint matrix for the parameter vector is

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

If V is the null space of $\boldsymbol{A}$, $\mathcal{C} = \{\boldsymbol{\beta} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{\beta} \geq \mathbf{0}\}$, and $\Omega = \mathcal{C} \cap V^{\perp}$, find the $\boldsymbol{\delta}$ vectors that are the edges of Ω.

6. Consider the balanced one-way analysis of variance (ANOVA) model

$$y_{ji} = \beta_j + \varepsilon_{ji} \text{ for } i = 1, \ldots, n, \text{ and } j = 1, \ldots, k, \tag{1}$$

where the ε_{ji} are *iid* standard normal random variables. Suppose $k = 3$ and *a priori* information allows us to assume ordered means: $\beta_1 \leq \beta_2 \leq \beta_3$. Note that the data can be summarized by the group means $\bar{\boldsymbol{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_3)'$, which is also the unconstrained estimate $\tilde{\boldsymbol{\beta}}$. How many sectors are there? For each sector, describe the data vectors $\bar{\boldsymbol{y}}$ that fall in that sector. If the true $\boldsymbol{\beta} = (1, 1, 1)'$, determine the probabilities that $\bar{\boldsymbol{y}}$ falls in each sector, using calculations or computer simulations.