Homework #1

due Wednesday, September 7th

STAT 740: CONSTRAINED ESTIMATION AND INFERENCE

FALL 2011

1. For the simple linear regression problem $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, let $\tilde{\beta}_0$ and $\tilde{\beta}_1$ be the unconstrained least-squares estimates. Now consider the constraint $\beta_1 \geq 0$. Prove that if $\tilde{\beta}_1 < 0$, the constrained least squares estimate is $\hat{\beta}_1 = 0$ and $\hat{\beta}_0 = \bar{y}$.

2. Suppose V_1 and V_2 are linear subspaces of $\mathbb{R}^n$ such that $V_1 \perp V_2$. Let V be the set of points in $\mathbb{R}^n$ that consist of $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2$ for all real numbers a_1 and a_2. (a) show V is a linear subspace of $\mathbb{R}^n$ and (b) show that the projection of $\boldsymbol{y} \in \mathbb{R}^n$ onto V is the sum of the projections of $\boldsymbol{y}$ onto V_1 and V_2.

3. If $\mathcal{S} \subseteq \mathbb{R}^n$ is a closed convex cone and $\boldsymbol{y} \in \mathbb{R}^n$, then $\hat{\boldsymbol{\theta}}$ minimizes $\parallel \boldsymbol{y} - \boldsymbol{\theta} \parallel^2$ over $\boldsymbol{\theta} \in \mathcal{S}$ if and only if

$$\langle \boldsymbol{y} - \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}} \rangle = 0 \ \text{ and } \ \langle \boldsymbol{y} - \hat{\boldsymbol{\theta}}, \boldsymbol{\theta} \rangle \leq 0, \ \text{ for all } \ \boldsymbol{\theta} \in \mathcal{S}.$$

   You can start with the KKT conditions for minimization over a convex set.

4. Consider the Kentucky Derby data set posted on the class website. Plot winning speed against year of race, and get a least-squares quadratic fit to the scatterplot. Note that it looks like the winning speeds have, on average, been decreasing since about 1989. Suppose a racing fan believes *a priori* that the average speed must be increasing in time, and would like to constrain the fit to be non-decreasing. What is the constraint matrix? Using a least-squares criterion, fit a parabola that is constrained to be increasing, using the cone projection algorithm `coneproj` that is posted on the class website. (You will have to transform the least-squares problem to a cone projection, then transform back to get the regression coefficients.)

5. Let $\mathcal{C}^o = \{\boldsymbol{\rho} \in \mathbb{R}^n : -\boldsymbol{\Delta}'\boldsymbol{\rho} \geq \boldsymbol{0}\}$, where the columns of the $n \times m$ matrix $\boldsymbol{\Delta}$ are the edge vectors $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_m$ calculated using a full row rank constraint matrix $\boldsymbol{A}$. Show that the polar cone could be defined as $\Omega^o = \mathcal{C}^o \cap V^\perp$.