Change–point estimation in ARCH models *

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Abstract

The paper studies the change–point problem and the cross–covariance function for ARCH models. Bounds for the cross-covariance function are derived and explicit formulas in special cases are obtained. Consistency of a CUSUM type change–point estimator is proved and its rate of convergence is established. A Hájek-Rényi type inequality is proved. The results are obtained under weak moment assumptions.

Keywords: ARCH model, Change–point estimation, Cross–covariance function

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1 Introduction

This paper studies ARCH models defined by equations

$$r_k = \sigma_k \varepsilon_k, \qquad \sigma_k^2 = a + \sum_{j=1}^{\infty} b(j) r_{k-j}^2, \tag{1.1}$$

where the ε_k are iid errors and a and the $b(j)$ are non–negative constants. Since their introduction by Engle (1982), ARCH type models have become perhaps the most popular and extensively studied financial econometric models. The literature on the subject is so vast that we will restrict ourselves to directing the reader to fairly comprehensive reviews of Bollerslev *et al.* (1992) and Shephard (1996). Model (1.1) is suitable for series r_k such that the observations are uncorrelated but exhibit "clusters of volatility".

In this paper we study the estimation of a change–point in the model (1.1). Note first that if the ε_k have mean zero, unit variance and the sequence $\{r_k\}$ is weakly stationary, then

$$\mathrm{Var}[r_k] = a/(1 - \sum_{j=1}^{\infty} b(j)). \tag{1.2}$$

Suppose now that the parameters a and $b(j)$ change at an *unknown* point k^* in such a way that the variance given by (1.2) changes. We are interested in estimating the change–point k^*. This problem is different from the problem of testing for structural changes where k^* is assumed to be *known* and one tests whether a change in model structure has taken place at k^*. Hsu (1977), Inclán and Tiao (1994), Chen and Gupta (1997) studied a simpler variance-change model $r_k = \tilde{\sigma}_k \varepsilon_k$, where $\tilde{\sigma}_k^2 = a_1$, if $1 \le k \le k^*$, and $\tilde{\sigma}_k^2 = a_2$, if $k^* < k \le n$, and where $\{\varepsilon_k\}$ is a sequence of independent standard normal random variables. Horváth (1993) considered tests against alternatives allowing changes both in means and variances of a sequence of independent normal random variables, whereas estimators and tests for change in variance under weak moment conditions were considered in Gombay *et al.* (1996).

By contrast, the change–point problem for linear sequences has received a great deal of attention. Csörgő and Horváth (1988), (1997) and Brodsky and Darkhovsky (1993) focus mainly on sequences of independent observations. Recently there has, however, been increasing interest in dependent data, see Antoch *et al.* (1997), Bai (1994), Bai and Perron (1998), Davis *et al.* (1995), Giraitis *et al.* (1996), Giraitis and Leipus (1990), (1992), Horváth and Kokoszka (1997), Leipus (1994), among others. The present paper studies processes which exhibit no *linear* dependence, but are characterized by the dependence of the conditional variance on past observations. This is known in econometrics as "dependence in volatility".

We study a CUSUM type estimator $\hat{k}$ of k^* defined as follows:

$$\hat{k} = \min\{k : |R_k| = \max_{1 \le j \le n} |R_j|\}, \tag{1.3}$$

where

$$R_k = \frac{k(n-k)}{n^2} \left(\frac{1}{k} \sum_{j=1}^{k} r_j^2 - \frac{1}{n-k} \sum_{j=k+1}^{n} r_j^2 \right). \tag{1.4}$$

Figure 1 below illustrates how the method works. The left panel shows a simulated realization $\{r_k, \ 1 \le k \le 1000\}$ of an ARCH(1) sequence with conditional variance $a_1 + b r_{k-1}^2$ for $k \le 700$ and $a_2 + b r_{k-1}^2$ for $k > 700$. We set $a_1 = 1, a_2 = 1.3, b = 0.1$ and used standard normal innovations ε_k. It is impossible to tell by eye if and where a change in variance occurred. The right panel shows the corresponding sequence R_k. The maximum is attained at $\hat{k} = 699$.

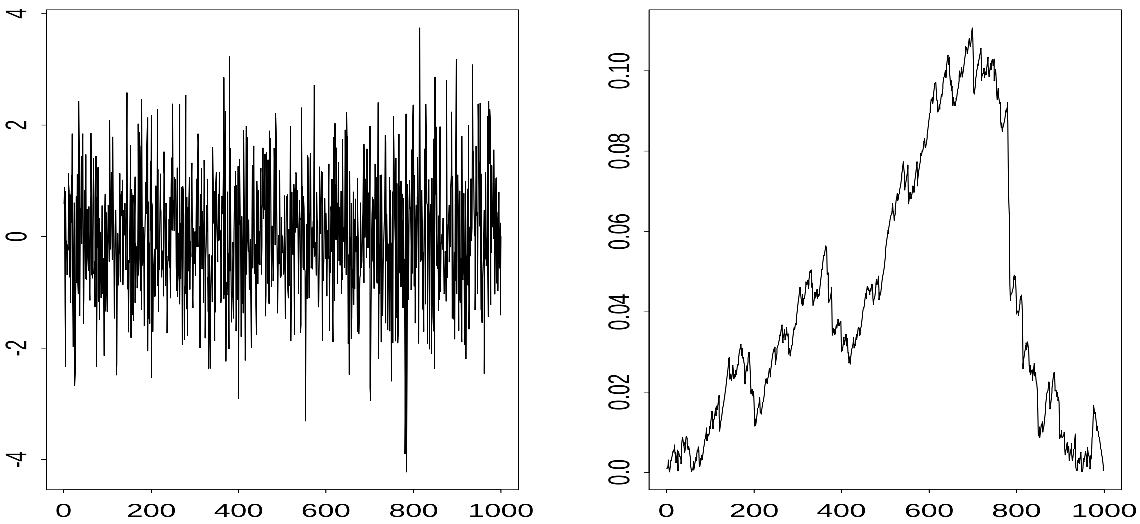

FIG. 1. A simulated realization of an ARCH(1) sequence with change–point at $k^* = 700$ and the corresponding sequence R_k.

Despite the standard form of the estimator $\hat{k}$, proving its consistency is not easy. For analogous estimators of a change–point in mean of a linear process $X_k = \mu_k + e_k$ the dependence structure of the errors e_k must be known. For ARCH models, we must be able to say something about the cross–covariance function of two ARCH sequences driven by the same innovations ε_k. This issue, which takes up the bulk of the paper, is addressed in Section 3. We believe that the results obtained in Section 3 are also of independent interest, as the study the covariance function of ARCH type sequences has occupied an important place in their theory. The method we propose is based on a Volterra series type expansion of the random variables r_k^2 derived in Section 2. It is different from methods based on matrix manipulations which go back to Engle (1982) and are applicable only to finite order ARCH and GARCH models, see He and Teräsvirta (1997) and references therein.

Another difficulty lies in the complex probabilistic structure of the sequence $\{r_k^2\}$, which does not fall into any well-studied class of stochastic processes for which appropriate inequalities are available. Thus, we had to prove in Section 4 a rather general Hájek-Rényi type inequality which we believe is also of independent interest.

Section 5 contains the proof of the consistency of the estimator $\hat{k}$, which is fairly standard once the tools developed in Sections 3 and 4 are available. The rate of convergence of the estimator is established in Section 6. A small simulation study and a data example are discussed in Section 7.

2 Stationarity of an ARCH(∞) sequence

In this section we recall the definition of an ARCH(∞) sequence and give a simple sufficient condition for ARCH(∞) equations (2.1) and (2.2) to have a weakly stationary solution. As far as we know, condition (2.3) below is new. Our method of proof is also different from the methods developed in Engle (1982) and Bollerslev (1986) and a large number of subsequent publications. It provides a closed form solution to the equations (2.1) and (2.2) and thus extends a similar result for ARCH(1) models, see e.g. Theorem 8.4.9 of Embrechts *et al.* (1997). Note also that we do not assume that the ε_k in Definition 2.1 below are Gaussian but merely that they have finite fourth moment. Thus, condition (2.3) is also a sufficient condition for the existence of the fourth moments of the solution of equations (2.1) and (2.2). Necessary and sufficient conditions have recently been established by He and Teräsvirta (1997) for finite order GARCH models.

We focus here on the squares of what is commonly referred to as an ARCH(∞) sequence $r_k = \sigma_k \varepsilon_k$ with the conditional variances σ_k^2 defined by (2.2); the sequence of *squares*, $\{r_k^2\}$, is called in the sequel an ARCH process.

DEFINITION 2.1 We say that a random sequence $\{r_k^2,\ k \in \mathbf{Z}\}$ satisfies ARCH(∞) equations if there exists a sequence of zero mean iid random variables $\{\varepsilon_k,\ k \in \mathbf{Z}\}$ with $E\varepsilon_k^2 = 1$ and $E\varepsilon_k^4 = \lambda < \infty$, such that

$$r_k^2 = \sigma_k^2 \varepsilon_k^2, \tag{2.1}$$

$$\sigma_k^2 = a + \sum_{j=1}^{\infty} b(j) r_{k-j}^2, \tag{2.2}$$

where $a \geq 0$, $b(j) \geq 0$, $j = 1, 2, \ldots$

THEOREM 2.1 *Consider the quantities $B = \sum_{k=1}^{\infty} b(k)$ and $\lambda = E\varepsilon_k^4$ as in Definition 2.1. If the condition*

$$\lambda B^2 < 1 \tag{2.3}$$

is satisfied, then ARCH(∞) equations (2.1) and (2.2) have a weakly stationary solution $\{r_k^2\}$ given by

$$r_k^2 = aY_k, \tag{2.4}$$

where

$$Y_k = \varepsilon_k^2 \sum_{l=0}^{\infty} M(k,l) \tag{2.5}$$

and

$$M(k,l) = \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{m=1}^{l} \varepsilon_{k-j_1-\ldots-j_m}^2 b(j_m) \right), \quad l > 0, \quad M(k,0) = 1. \tag{2.6}$$

In the sequel we adopt the convention that $\prod_l^{l-i} a_j = 1$, $\sum_m^{m-i} a_j = 0$ for $i > 0$, and denote $a \wedge b := \min\{a, b\}$.

PROOF: We will show that $\{aY_k\}$ is a second-order stationary sequence satisfying ARCH(∞) equations (2.1), (2.2).

Assumption $E\varepsilon_1^2 = 1$ and the Cauchy inequality imply that $\lambda \geq 1$ and thus by (2.3) $B < 1$. Since all factors of the product $\prod_{m=1}^{l} \varepsilon_{k-j_1-\ldots-j_m}^2$ in (2.6) are independent, we obtain

$$EM(k,l) = \sum_{j_1,\ldots,j_l=1}^{\infty} \prod_{m=1}^{l} b(j_m) = B^l$$

and

$$EY_k = \sum_{l=0}^{\infty} EM(k,l) = \sum_{l=0}^{\infty} B^l = \frac{1}{1-B} < \infty.$$

Therefore $M(k,l)$ and Y_k are well defined: for any k and l, $M(k,l) < \infty$ a.s. and $Y_k < \infty$ a.s. Using the recursive equality

$$M(k,l) = \sum_{j=1}^{\infty} b(j)\varepsilon_{k-j}^2 M(k-j, l-1)$$

it is easy to verify that $\{aY_k\}$ is a solution to (2.1), (2.2). Next we will show that the series $\sum_{l=1}^{\infty} M(k,l)$ converges in L^2. Fatou's lemma will then imply that the a.s. and the L^2 limits of the partial sums coincide. To verify L^2-convergence we use the Cauchy criterion. Write

$$E|\sum_{n_1<l\le n_2} M(k,l)|^2 = \sum_{n_1<l,l'\le n_2} EM(k,l)M(k,l'), \tag{2.7}$$

where

$$EM(k,l)M(k,l')$$
$$= \sum_{j_1,\ldots,j_l=1}^{\infty} \sum_{j'_1,\ldots,j'_{l'}=1}^{\infty} \left[\left(\prod_{m=1}^{l} b(j_m)\right)\left(\prod_{m'=1}^{l'} b(j'_{m'})\right) E\left(\prod_{m=1}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} \prod_{m'=1}^{l'} \varepsilon^2_{k-j'_1-\ldots-j'_{m'}}\right)\right]$$
$$\le \sum_{j_1,\ldots,j_l=1}^{\infty} \sum_{j'_1,\ldots,j'_{l'}=1}^{\infty} \left[\left(\prod_{m=1}^{l} b(j_m)\right)\left(\prod_{m'=1}^{l'} b(j'_{m'})\right)\left(E\prod_{m=1}^{l} \varepsilon^4_{k-j_1-\ldots-j_m}\right)^{1/2}\left(E\prod_{m'=1}^{l'} \varepsilon^4_{k-j'_1-\ldots-j'_{m'}}\right)^{1/2}\right]$$
$$= \lambda^{(l+l')/2} B^{l+l'}.$$

This combined with (2.3) and (2.7) yields

$$E|\sum_{n_1<l\le n_2} M(k,l)|^2 \to 0, \text{ as } n_1, n_2 \to \infty.$$

To verify that the sequence r_k^2 defined by (2.4) is weakly stationary, observe that

$$EY_k = 1/(1-B) = \sum_{l=0}^{\infty} \sum_{j_1,\ldots,j_l=1}^{\infty} \prod_{m=1}^{l} b(j_m), \tag{2.8}$$

and hence

$$\mathrm{Cov}(Y_i, Y_{i+k}) = E(Y_i Y_{i+k}) - EY_i EY_{i+k} \tag{2.9}$$
$$= \sum_{l,l'=0}^{\infty} \sum_{j'_1,\ldots,j'_{l'}=1}^{\infty} \left(\prod_{m'=1}^{l'} b(j'_{m'})\right) \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{m=1}^{l} b(j_m)\right) E(\prod_{m'=0}^{l'} \varepsilon^2_{i-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{i+k-j_1-\ldots-j_m} - 1)$$
$$= \mathrm{Cov}(Y_0, Y_k).$$

□

3 The cross–covariance function of ARCH(∞) sequences

In order to find the *correlation* function (of the squares) of an ARCH(p) or GARCH(p,q) sequence with *finite* p and q equalities analogous to Yule–Walker equations have been used in the existing literature. For example, for the ARCH(1) sequence

$$r_k^2 = (a + br_{k-1}^2)\varepsilon_k^2 \tag{3.1}$$

multiplying both sides of (3.1) by r_0^2, taking expectations and using the relation $Er_k^2 = a/(1-b)$ we obtain $\mathrm{Cov}(r_0^2, r_k^2) = b^k \mathrm{Var}\, r_0^2$. However, computing the variance $\mathrm{Var}\, r_0^2$ is not easy and requires a representation of r_0^2 in terms of the ε_k^2.

Similarly, in GARCH(p,q) case, one can derive Yule–Walker type equalities (see Bollerslev (1986)) which can be solved when the first $q+1$ covariances are known.

In this section we focus on the cross–covariance function of two ARCH(∞) sequences $\{r_{1,k}^2, k \in \mathbf{Z}\}$ and $\{r_{2,k}^2, k \in \mathbf{Z}\}$ driven by the same sequence $\{\varepsilon_k\}$ of iid random variables with $E\varepsilon_1 = 0$, $E\varepsilon_1^2 = 1$ and $E\varepsilon_1^4 = \lambda < \infty$ i.e.

$$r_{i,k}^2 = \sigma_{i,k}^2 \varepsilon_k^2, \quad \sigma_{i,k}^2 = a_i + \sum_{j=1}^{\infty} b_i(j) r_{i,k-j}^2, \quad i = 1, 2, \tag{3.2}$$

where $a_i \geq 0$, and $\{b_i(j)\}$, $i = 1, 2$, are two sequences of non–negative numbers.

Our results are valid for "short memory" ARCH(∞) models, i.e. we assume in the sequel that the $b_i(j)$ decay exponentially fast. More specifically, we suppose in the remainder of the paper that the following assumption holds:

ASSUMPTION 3.1 The coefficients $b(j)$ in Definition 2.1 satisfy

$$b(j) \leq \beta\alpha^j \tag{3.3}$$

for some $0 \leq \alpha < 1$ and $\beta \geq 0$ such that

$$\lambda \left(\frac{\beta\alpha}{1-\alpha} \right)^2 < 1. \tag{3.4}$$

Clearly, condition (3.4) implies (2.3).

In the proof of Theorem 3.1 and in the sequel we will extensively use the quantities

$$B_i(k) := b_i(k) + b_i(k+1) + \ldots \quad \text{and} \quad B_i := B_i(1), \ i = 1, 2.$$

We now formulate the main theorem of this section.

THEOREM 3.1 *If the sequences $\{r_{1,k}^2\}$ and $\{r_{2,k}^2\}$ satisfy Assumption 3.1, then*

$$0 \leq \mathrm{Cov}(r_{1,k}^2, r_{2,k'}^2) \leq a_1 a_2 C_1 \alpha^{|k'-k|} (1+\beta)^{|k'-k|}, \tag{3.5}$$

where

$$C_1 = \frac{\beta(\lambda - 1)}{(1+\beta)(1-\alpha)(1-B_1)(1-B_2)(1-B_1B_2\lambda)}. \tag{3.6}$$

Note that since $\lambda \geq 1$, condition (3.4) implies $\alpha(1+\beta) < 1$.

PROOF: Set $r_{1,k}^2 = a_1 Y_k^{(1)}$, $r_{2,k}^2 = a_2 Y_k^{(2)}$, where the $Y_k^{(i)}$ are defined by (2.5) with corresponding parameters $b_i(1), b_i(2), \ldots$ Note that

$$\mathrm{Cov}(Y_k^{(1)}, Y_{k'}^{(2)}) = \tag{3.7}$$

$$= \sum_{l,l'=0}^{\infty} \sum_{j'_1,\ldots,j'_{l'}=1}^{\infty} \left(\prod_{m'=1}^{l'} b_2(j'_{m'}) \right) \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{m=1}^{l} b_1(j_m) \right) E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1).$$

Assume first that $k > k'$. Then the products $\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}}$ and $\prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m}$ in (3.7) are independent if $l = 0$ or $j_1 + \ldots + j_l < k - k'$. Thus we can write

$$\mathrm{Cov}(Y_k^{(1)}, Y_{k'}^{(2)}) \tag{3.8}$$

$$= \sum_{\substack{l=1 \\ l'=0}}^{\infty} \sum_{j'_1,\ldots,j'_{l'}=1}^{\infty} \left(\prod_{m'=1}^{l'} b_2(j'_{m'}) \right) \sum_{\substack{j_1,\ldots,j_l=1 \\ j_1+\ldots+j_l \geq k-k'}}^{\infty} \left(\prod_{m=1}^{l} b_1(j_m) \right) E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1).$$

We now split the last sum in (3.8) into a sum over $j_1 + \ldots + j_{l-1} \geq k - k'$ and a sum over $j_1 + \ldots + j_{l-1} < k - k'$. This procedure is carried on until the second sum is split into l sums $S_1, \ldots, S_l$, c.f. (3.9) and (3.10) below, such that S_i can be appropriately bounded above, c.f. (3.16). The latter bound is then used in (3.17) and (3.18), which lead to (3.5). Write then

$$\sum_{\substack{j_1,\ldots,j_l=1 \\ j_1+\ldots+j_l \geq k-k'}}^{\infty} \left(\prod_{m=1}^{l} b_1(j_m) \right) E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1)$$

$$= \sum_{\substack{j_1,\ldots,j_l=1 \\ j_1+\ldots+j_{l-1} \geq k-k'}}^{\infty} \left(\prod_{m=1}^{l} b_1(j_m) \right) E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1)$$

$$+ \sum_{\substack{j_1,\ldots,j_{l-1}=1 \\ j_1+\ldots+j_{l-1} < k-k'}}^{\infty} \left(\prod_{m=1}^{l-1} b_1(j_m) \right) \sum_{j_l=k-k'-j_1-\ldots-j_{l-1}}^{\infty} b_1(j_l) E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1)$$

$$=: T + S_1. \tag{3.9}$$

For the term T consider analogously the cases $j_1 + \ldots + j_{l-2} \geq k - k'$ and $j_1 + \ldots + j_{l-2} < k - k'$:

$$T = \sum_{\substack{j_1,\ldots,j_l=1 \\ j_1+\ldots+j_{l-2} \geq k-k'}}^{\infty} \left(\prod_{m=1}^{l} b_1(j_m) \right) E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1)$$

$$+ \sum_{\substack{j_1,\ldots,j_{l-2}=1 \\ j_1+\ldots+j_{l-2} < k-k'}}^{\infty} \left(\prod_{m=1}^{l-2} b_1(j_m) \right) \sum_{j_{l-1}=k-k'-j_1-\ldots-j_{l-2}}^{\infty} b_1(j_{l-1}) \sum_{j_l=1}^{\infty} b_1(j_l)$$

$$\times E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1).$$

Continuing the above procedure, we obtain

$$T = S_2 + \ldots + S_l, \tag{3.10}$$

where

$$S_i = \sum_{\substack{j_1,\ldots,j_{l-i}=1 \\ j_1+\ldots+j_{l-i}<k-k'}}^{\infty} \left(\prod_{m=1}^{l-i} b_1(j_m)\right) \sum_{j_{l-i+1}=k-k'-j_1-\ldots-j_{l-i}}^{\infty} b_1(j_{l-i+1}) \sum_{j_{l-i+2}=1}^{\infty} b_1(j_{l-i+2}) \ldots \sum_{j_l=1}^{\infty} b_1(j_l)$$

$$E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=0}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1).$$

For the term S_l we can write

$$S_l \le \sum_{j_1=k-k'}^{\infty} b_1(j_1) \sum_{j_2=1}^{\infty} b_1(j_2) \ldots \sum_{j_l=1}^{\infty} b_1(j_l)(\lambda^{(l'+1)\wedge l} - 1) \tag{3.11}$$

$$= B_1(k-k')\; B_1^{l-1}\; (\lambda^{(l'+1)\wedge l} - 1).$$

For the terms S_i $(i = 1, \ldots, l-1)$ we get

$$S_i = \sum_{\substack{j_1,\ldots,j_{l-i}=1 \\ j_1+\ldots+j_{l-i}<k-k'}}^{\infty} \left(\prod_{m=1}^{l-i} b_1(j_m)\right) \sum_{j_{l-i+1}=k-k'-j_1-\ldots-j_{l-i}}^{\infty} b_1(j_{l-i+1}) \sum_{j_{l-i+2}=1}^{\infty} b_1(j_{l-i+2}) \ldots \sum_{j_l=1}^{\infty} b_1(j_l) \tag{3.12}$$

$$\times E(\prod_{m'=0}^{l'} \varepsilon^2_{k'-j'_1-\ldots-j'_{m'}} \prod_{m=l-i+1}^{l} \varepsilon^2_{k-j_1-\ldots-j_m} - 1).$$

$$\le \sum_{\substack{j_1,\ldots,j_{l-i}=1 \\ j_1+\ldots+j_{l-i}<k-k'}}^{\infty} \left(\prod_{m=1}^{l-i} b_1(j_m)\right) \sum_{j_{l-i+1}=k-k'-j_1-\ldots-j_{l-i}}^{\infty} b_1(j_{l-i+1}) \ldots \sum_{j_l=1}^{\infty} b_1(j_l)(\lambda^{(l'+1)\wedge i} - 1)$$

$$= (\lambda^{(l'+1)\wedge i} - 1)B^{i-1} \sum_{\substack{j_1,\ldots,j_{l-i}=1 \\ j_1+\ldots+j_{l-i}<k-k'}}^{\infty} \left(\prod_{m=1}^{l-i} b_1(j_m)\right) \sum_{j_{l-i+1}=k-k'-j_1-\ldots-j_{l-i}}^{\infty} b_1(j_{l-i+1}).$$

By the assumption $b_i(j) \le \beta\alpha^j$

$$\sum_{j_{l-i+1}=k-k'-j_1-\ldots-j_{l-i}}^{\infty} \left(\prod_{m=1}^{l-i} b_1(j_m)\right) b_1(j_{l-i+1}) \le \beta^{l-i+1} \frac{\alpha^{k-k'}}{1-\alpha}. \tag{3.13}$$

Thus for $i = 1, \ldots l-1$

$$S_i \le (\lambda^{(l'+1)\wedge i} - 1)B_1^{i-1}\beta^{l-i} A_{k-k'} \sum_{\substack{j_1,\ldots,j_{l-i}=1 \\ j_1+\ldots+j_{l-i}<k-k'}}^{\infty} 1, \tag{3.14}$$

where

$$A_k := \frac{\beta\alpha^k}{1-\alpha}.$$

Since

$$\sum_{\substack{j_1,\dots,j_l=1\\ j_1+\dots+j_l=m}}^{\infty} 1 = \binom{m-1}{l-1},$$

we have

$$\sum_{\substack{j_1,\dots,j_{l-i}=1\\ j_1+\dots+j_{l-i}<k-k'}}^{\infty} 1 = \sum_{j=0}^{k-k'-l+i-1} \binom{j+l-i-1}{j} 1_{\{l-i<k-k'\}}. \tag{3.15}$$

By (3.14) and (3.15) it follows that for $i = 1, \dots l-1$

$$S_i \le (\lambda^{(l'+1)\wedge i} - 1) B_1^{i-1} \beta^{l-i} A_{k-k'} \sum_{j=0}^{k-k'-l+i-1} \binom{j+l-i-1}{j} 1_{\{l-i<k-k'\}}. \tag{3.16}$$

Substitute now these estimates to (3.10), (3.9) and (3.8) and write

$$\begin{aligned}\mathrm{Cov}(Y_k^{(1)}, Y_{k'}^{(2)}) &\le \sum_{\substack{l=1\\ l'=0}}^{\infty} \sum_{j'_1,\dots,j'_{l'}=1}^{\infty} \left(\prod_{m'=1}^{l'} b_2(j'_{m'}) \right) S_l + \sum_{\substack{l=1\\ l'=0}}^{\infty} \sum_{j'_1,\dots,j'_{l'}=1}^{\infty} \left(\prod_{m'=1}^{l'} b_2(j'_{m'}) \right) \sum_{i=1}^{l-1} S_i \\ &=: s_1 + s_2.\end{aligned}$$

Consider first the term s_1

$$s_1 = \sum_{l'=0}^{\infty} B_2^{l'} \sum_{l=1}^{\infty} S_l \le B_1(k-k') \left[\sum_{l'=0}^{\infty} B_2^{l'} \sum_{l=1}^{\infty} B_1^{l-1} \lambda^{(l'+1)\wedge l} - \sum_{l'=0}^{\infty} B_2^{l'} \sum_{l=1}^{\infty} B_1^{l-1} \right] \tag{3.17}$$

$$= B_1(k-k') \left[\sum_{l'=0}^{\infty} B_2^{l'} \sum_{l=1}^{l'+1} B_1^{l-1} \lambda^{l} + \sum_{l'=0}^{\infty} B_2^{l'} \sum_{l=l'+2}^{\infty} B_1^{l-1} \lambda^{l'+1} - (1-B_1)^{-1}(1-B_2)^{-1} \right]$$

$$= B_1(k-k') \left[\sum_{l=1}^{\infty} B_1^{l-1} \lambda^{l} \sum_{l'=l-1}^{\infty} B_2^{l'} + \sum_{l'=0}^{\infty} B_2^{l'} \lambda^{l'+1} \sum_{l=l'+2}^{\infty} B_1^{l-1} - (1-B_1)^{-1}(1-B_2)^{-1} \right]$$

$$= B_1(k-k') \left[\frac{\lambda}{(1-B_2)(1-B_1B_2\lambda)} + \frac{B_1\lambda}{(1-B_1)(1-B_1B_2\lambda)} - \frac{1}{(1-B_1)(1-B_2)} \right] =$$

$$= B_1(k-k') \; \frac{\lambda - 1}{(1-B_1)(1-B_2)(1-B_1B_2\lambda)}.$$

For the term s_2 we have by (3.16)

$$s_2 \le A_{k-k'} \sum_{\substack{l=1\\ l'=0}}^{\infty} \sum_{j'_1,\dots,j'_{l'}=1}^{\infty} \left(\prod_{m'=1}^{l'} b_2(j'_{m'}) \right) \tag{3.18}$$

$$\times \sum_{i=1}^{l-1} \beta^{l-i} (\lambda^{(l'+1)\wedge i} - 1) B_1^{i-1} \sum_{j=0}^{k-k'-l+i-1} \binom{j+l-i-1}{j} 1_{\{l-i<k-k'\}}$$

$$= A_{k-k'} \sum_{l'=0}^{\infty} B_2^{l'} \sum_{i=1}^{\infty} (\lambda^{(l'+1)\wedge i} - 1) B_1^{i-1} G_\beta(k-k'),$$

where

$$G_\beta(k) := \sum_{l=i+1}^{i+k-1} \beta^{l-i} \sum_{j=0}^{k-l+i-1} \binom{j+l-i-1}{j}$$

$$= \sum_{m=1}^{k-1} \beta^m \sum_{j=0}^{k-m-1} \binom{j+m-1}{j} = \sum_{m=1}^{k-1} \beta^m \binom{k-1}{m}.$$

It is easy to see that $G_\beta(1) = 0$, $G_\beta(2) = \beta$, ..., $G_\beta(k) = G_\beta(k-1) + \beta(\beta+1)^{k-2}$, $k \geq 2$ and hence

$$G_\beta(k) = (\beta+1)^{k-1} - 1.$$

Using (3.18) we now establish an upper bound for s_2 as follows:

$$s_2 \leq A_{k-k'} G_\beta(k-k') \tag{3.19}$$

$$\times \left[\sum_{l'=0}^{\infty} B_2^{l'} \sum_{i=1}^{l'} \lambda^i B_1^{i-1} + \sum_{l'=0}^{\infty} B_2^{l'} \sum_{i=l'+1}^{\infty} \lambda^{l'+1} B_1^{i-1} - \sum_{l'=0}^{\infty} B_2^{l'} \sum_{i=1}^{\infty} B_1^{i-1} \right]$$

$$= A_{k-k'} G_\beta(k-k') \left[\sum_{i=1}^{\infty} \lambda^i B_1^{i-1} \sum_{l'=i}^{\infty} B_2^{l'} + \sum_{l'=0}^{\infty} B_2^{l'} \lambda^{l'+1} \sum_{i=l'+1}^{\infty} B_1^{i-1} - \frac{1}{(1-B_1)(1-B_2)} \right]$$

$$= A_{k-k'} G_\beta(k-k') \left[\sum_{i=1}^{\infty} \lambda^i B_1^{i-1} (1-B_2)^{-1} B_2^i + \sum_{l'=0}^{\infty} B_2^{l'} \lambda^{l'+1} (1-B_1)^{-1} B_1^{l'} - \frac{1}{(1-B_1)(1-B_2)} \right]$$

$$= A_{k-k'} G_\beta(k-k') \left[\frac{\lambda B_2}{(1-B_2)(1-B_1B_2\lambda)} + \frac{\lambda}{(1-B_1)(1-B_1B_2\lambda)} - \frac{1}{(1-B_1)(1-B_2)} \right]$$

$$= \alpha^{k-k'} G_\beta(k-k') \; \frac{\beta(\lambda-1)}{(1-\alpha)(1-B_1)(1-B_2)(1-B_1B_2\lambda)}.$$

Applying now (3.17) and (3.19), we get

$$\mathrm{Cov}(Y_k^{(1)}, Y_{k'}^{(2)}) \leq B_1(k-k') \; \frac{\lambda-1}{(1-B_1)(1-B_2)(1-B_1B_2\lambda)} \tag{3.20}$$

$$+\alpha^{k-k'} G_\beta(k-k') \; \frac{\beta(\lambda-1)}{(1-\alpha)(1-B_1)(1-B_2)(1-B_1B_2\lambda)}$$

$$= \frac{\lambda-1}{(1-B_1)(1-B_2)(1-B_1B_2\lambda)} \; (B_1(k-k') + \frac{\beta}{1-\alpha} \alpha^{k-k'} G_\beta(k-k')).$$

The inequality $B_1(k-k') \leq \beta\alpha^{k-k'}(1-\alpha)^{-1}$ yields (3.5).

So far we have been assuming that $k > k'$ in order to ensure the independence of certain products in (3.7). The case $k' > k$ is analogous. If $k = k'$, observe first that setting $a_i = 1$ yields $Y_k^{(i)} = r_{i,k}^2$. Then the claim follows from the general identity

$$\mathrm{Cov}(r_{1,k}^2, r_{2,k}^2) = \sum_{j=1}^{\infty} b_1(j) \mathrm{Cov}(r_{1,k-j}^2, r_{2,k}^2), \tag{3.21}$$

which can be obtained by multiplying both sides of

$$r_{1,k}^2 = a_1 + \sum_{j=1}^{\infty} b_1(j) r_{1,k-j}^2$$

by $r_{2,k}^2$, taking expectations, and using the identity $Er_{i,k}^2 = a_i/(1 - B_i)$. Then use the relation

$$\mathrm{Cov}(r_{1,k-j}^2, r_{2,k}^2) = \mathrm{Cov}(r_{1,j}^2, r_{2,0}^2) \le C_1(\alpha(1+\beta))^j$$

and (3.5) with $k - k' = j > 0$. □

REMARK 3.1 Careful examination of the proof of Theorem 3.1 reveals that the Assumption 3.1 can be replaced by the condition

$$b_i(j')b_i(j) \le \beta_i b_i(j' + j), \quad i = 1, 2. \tag{3.22}$$

Using (3.22) in (3.13) instead of Assumption 3.1 we obtain the following inequalities

$$\mathrm{Cov}(Y_k^{(1)}, Y_{k'}^{(2)}) \le \begin{cases} \dfrac{\lambda - 1}{(1 - B_1)(1 - B_2)(1 - B_1 B_2 \lambda)} B_1(k - k')(1 + \beta_1)^{k-k'-1}, & \text{if} \quad k \ge k', \\ \dfrac{\lambda - 1}{(1 - B_1)(1 - B_2)(1 - B_1 B_2 \lambda)} B_2(k' - k)(1 + \beta_2)^{k'-k-1}, & \text{if} \quad k < k', \end{cases}$$

where, roughly speaking, the factors $\alpha^{|k-k'|}$ in (3.5) are replaced by the tails $B_i(|k - k'|)$.

REMARK 3.2 Replacing $Y_{k'}^{(2)}$ by $Y_k^{(1)}$ in (3.5), a bound on the covariance function of the sequence $\{Y_k\}$ defined by (2.5) is obtained. The rate $\alpha^k(1+\beta)^k$ is optimal under Assumption 3.1. This can be verified by setting $b_j = \beta\alpha^j$, which corresponds to considering a GARCH$(1,1)$ model as in Bollerslev (1988) who verified an analogous result for the *correlation* function of GARCH$(1,1)$ models with normal errors. Obtaining a formula for the *covariance* function of GARCH models is much more complicated, see He and Teräsvirta (1997).

REMARK 3.3 Using the technique developed in the proof of Theorem 3.1, it is very easy to verify that for the ARCH(1) sequence $r_k^2 = (a + br_{k-1}^2)\varepsilon_k^2$ or, equivalently, $r_k^2 = aY_k$, where $Y_k = \sum_{j=0}^{\infty} b^j \prod_{m=0}^{j} \varepsilon_{k-m}^2$,

$$\mathrm{Cov}(Y_0, Y_k) = Kb^k, \tag{3.23}$$

where

$$K = \frac{\lambda - 1}{(1 - b^2\lambda)(1 - b)^2}. \tag{3.24}$$

This follows from the equality

$$\mathrm{Cov}(Y_0, Y_k) = E \sum_{j,j'=0}^{\infty} b^{j+j'} \Big(\prod_{m=0}^{j} \varepsilon_{-m}^2 \prod_{m'=0}^{j'} \varepsilon_{k-m'}^2 - 1 \Big)$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{j} b^j b^{k+i} (\lambda^{i+1} - 1) + \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} b^j b^{k+i} (\lambda^{j+1} - 1).$$

We have assumed so far that the sequences $\{r_{1,k}^2\}$ and $\{r_{2,k}^2\}$ start infinitely far in the past. After some modifications our methods can, however, also be applied to sequences which start at some initial time point. To illustrate the idea, we focus here only on ARCH(1) models. Assume then that sequences $\{r_{1,k}^2, k = 0, 1, \ldots\}$ and $\{r_{2,k}^2, k = 0, 1, \ldots\}$ are defined as follows:

$$\begin{aligned} r_{i,k}^2 &= \sigma_{i,k}^2 \varepsilon_k^2, \\ \sigma_{i,k}^2 &= a_i + b_i r_{i,k-1}^2, \quad k \geq 1, \end{aligned} \tag{3.25}$$

where $a_i \geq 0$, $0 \leq b_i < 1$ and $b_i^2 \lambda < 1$, $i = 1, 2$, which is precisely condition (2.3) of Theorem 2.1. The initial values $r_{1,0}$ and $r_{2,0}$, such that $Er_{i,0}^4 < \infty$, are assumed to be independent of $\{\varepsilon_t\}$, but not necessary mutually independent. The following result gives a closed form formula for the cross–covariance function of the sequences $\{r_{1,k}^2\}$ and $\{r_{2,k}^2\}$ starting at $k = 0$. The covariance function of each sequence $\{r_{i,k}^2\}$ can be obtained by setting $V = 0$ in (3.26) and replacing $a_1 a_2$ by a_i^2.

THEOREM 3.2 *For the sequences $\{r_{1,k}^2\}$ and $\{r_{2,k}^2\}$ given in (3.25) their cross-covariance function is*

$$\operatorname{Cov}(r_{1,k}^2, r_{2,k'}^2) = \begin{cases} U \; b_2^{k'-k} + V \; (b_1\lambda)^k b_2^{k'} \;, & \text{if} \quad k \leq k', \\ U \; b_1^{k-k'} + V \; (b_2\lambda)^{k'} b_1^{k} \;, & \text{if} \quad k > k', \end{cases} \tag{3.26}$$

where

$$\begin{aligned} U &= \frac{a_1 a_2 (\lambda - 1)}{(1-b_1)(1-b_2)(1-b_1 b_2 \lambda)}, \\ V &= -\frac{a_1 a_2 (1 - b_1 b_2)\lambda}{(1-b_1)(1-b_2)(1-b_1 b_2 \lambda)} + E(r_{1,0}^2 r_{2,0}^2). \end{aligned} \tag{3.27}$$

The proof is given in the Appendix.

4 A Hájek-Rényi type inequality

In this section we prove a Hájek-Rényi type inequality, i.e. an inequality for the maximum of weighted sums $c_k|X_1 + \ldots + X_k|$, where $X_1, \ldots, X_n$ are *any* finite variance random variables. The result is stated in Theorem 4.1 below. In the proof we will need an elementary inequality, which we now state as a lemma, as we could not find any reference to it.

LEMMA 4.1 *For any random variables $Y_1, \ldots, Y_n$ and events $A = \{\max_{1 \leq k \leq n} Y_k > \epsilon\}$, $D_k = \{Y_1 \leq \epsilon, \ldots, Y_k \leq \epsilon\}$*

$$\epsilon \mathbf{1}_A \leq Y_1 + \sum_{k=1}^{n-1} (Y_{k+1} - Y_k) \mathbf{1}_{D_k} - Y_n \mathbf{1}_{D_n}. \tag{4.1}$$

PROOF: Introduce the events $E_1 = \{Y_1 > \epsilon\}$, $E_i = \{Y_1 \leq \epsilon, \ldots, Y_{i-1} \leq \epsilon, Y_i > \epsilon\}$, $i = 2, \ldots, n$. Obviously

$$\epsilon \mathbf{1}_A = \epsilon \sum_{i=1}^n \mathbf{1}_{AE_i} = \sum_{i=1}^n \epsilon \mathbf{1}_{E_i} \leq \sum_{i=1}^n Y_i \mathbf{1}_{E_i}.$$

Applying Abel's transformation we can write

$$\sum_{i=1}^n Y_i \mathbf{1}_{E_i} = \sum_{i=1}^{n-1} U_i (Y_i - Y_{i+1}) + U_n Y_n, \tag{4.2}$$

where

$$U_i = \mathbf{1}_{E_1} + \ldots + \mathbf{1}_{E_i} = 1 - \mathbf{1}_{D_1} + \mathbf{1}_{D_1} - \mathbf{1}_{D_2} + \ldots + \mathbf{1}_{D_{i-1}} - \mathbf{1}_{D_i} = 1 - \mathbf{1}_{D_i}.$$

Thus by (4.2)

$$\begin{aligned}\sum_{i=1}^{n} Y_i \mathbf{1}_{E_i} &= \sum_{i=1}^{n-1}(1-\mathbf{1}_{D_i})(Y_i - Y_{i+1}) + (1-\mathbf{1}_{D_n})Y_n \\ &= Y_1 + \sum_{i=1}^{n-1}(Y_{i+1}-Y_i)\mathbf{1}_{D_i} - Y_n\mathbf{1}_{D_n}.\end{aligned}$$

□

THEOREM 4.1 *Let $X_1, \ldots, X_n$ be any random variables with finite second moments and $c_1, \ldots, c_n$ be any non–negative constants. Then*

$$\begin{aligned}\epsilon^2 P\{\max_{m\le k\le n} c_k |\sum_{j=1}^{k} X_j| > \epsilon\} &\le c_m^2 \sum_{i,j=1}^{m} E(X_iX_j) + \sum_{k=m}^{n-1} |c_{k+1}^2 - c_k^2| \sum_{i,j=1}^{k} E(X_iX_j) \\ &+2\sum_{k=m}^{n-1} c_{k+1}^2 E(|X_{k+1}|\,|\sum_{j=1}^{k} X_j|) + \sum_{k=m}^{n-1} c_{k+1}^2 E X_{k+1}^2. \qquad (4.3)\end{aligned}$$

PROOF: Applying (4.1) with $Y_k = c_k^2 |\sum_{j=1}^{k} X_j|^2$ we get

$$\epsilon^2 P\{\max_{m\le k\le n} c_k^2 |\sum_{j=1}^{k} X_j|^2 > \epsilon^2\} \le c_m^2 E|\sum_{j=1}^{m} X_j|^2 + E\sum_{k=m}^{n-1}(c_{k+1}^2|\sum_{j=1}^{k+1} X_j|^2 - c_k^2|\sum_{j=1}^{k} X_j|^2)\mathbf{1}_{D'_k},$$

where $D'_k = \{Y_m \le \epsilon^2, \ldots, Y_k \le \epsilon^2\}$. Each summand on the right-hand side can be estimated as follows:

$$\begin{aligned}&(c_{k+1}^2|\sum_{j=1}^{k+1} X_j|^2 - c_k^2|\sum_{j=1}^{k} X_j|^2)\mathbf{1}_{D'_k} \qquad (4.4)\\ &\le \Big(c_{k+1}^2|\sum_{j=1}^{k} X_j|^2 + 2c_{k+1}^2|\sum_{j=1}^{k} X_j||X_{k+1}| + c_{k+1}^2 X_{k+1}^2 - c_k^2|\sum_{j=1}^{k} X_j|^2\Big)\mathbf{1}_{D'_k} \\ &\le |c_{k+1}^2 - c_k^2|\,|\sum_{j=1}^{k} X_j|^2 + 2c_{k+1}^2|\sum_{j=1}^{k} X_j|\,|X_{k+1}| + c_{k+1}^2 X_{k+1}^2,\end{aligned}$$

which yields (4.3). □

5 Consistency of the change–point estimator

Assume that $\{r_{1,k}^2, k \in \mathbf{Z}\}$ and $\{r_{2,k}^2, k \in \mathbf{Z}\}$ are two ARCH(∞) sequences defined by (3.2), where $a_i \ge 0$, and $\{b_i(j)\}$ are sequences of non–negative numbers satisfying Assumption 3.1.

Suppose we observe a sample $r_1^2, \ldots, r_n^2$ from the model

$$r_k^2 = \begin{cases} r_{1,k}^2\ , & \text{if} \quad 1 \le k \le k^*, \\ r_{2,k}^2\ , & \text{if} \quad k^* < k \le n, \end{cases} \qquad (5.1)$$

where k^* is an *unknown* change–point. Let $k^* = \tau^* n$ and assume that $0 < \tau^* < 1$. Obviously, the "disordered" sequence $\{r_k^2\}$ is no longer stationary. For example, for the ARCH(1) model such that $a_1 \neq a_2$ and $b_1 = b_2 = b$ (c.f. (3.25)), the covariance function has the form

$$\mathrm{Cov}(r_k^2, r_{k'}^2) = \begin{cases} a_1^2\ K\ b^{|k'-k|} &, \text{ if } \ 1 \le k, k' \le k^*, \\ a_2^2\ K\ b^{|k'-k|} &, \text{ if } \ k^* < k, k' \le n, \\ a_1 a_2\ K\ b^{|k'-k|} &, \quad \text{otherwise}, \end{cases}$$

where K is given by (3.24). Interestingly, the correlation function of the "disordered" sequence is $b^{|k'-k|}$ and does not reflect the existence of non-stationarity. Therefore in the following we focus on the covariance function.

For ease of reference we now state a simple consequence of Theorem 3.1 which follows from the relation $r_{i,k}^2 = a_i Y_k$ with Y_k given by (2.5).

COROLLARY 5.1 *Suppose* (3.2), (5.1) *and Assumption 3.1 hold. Then*

$$0 \le \mathrm{Cov}(r_k^2, r_{k'}^2) \le \begin{cases} a_1^2\ C_1'\ (\alpha(1+\beta))^{|k'-k|} &, \text{ if } \ 1 \le k, k' \le k^*, \\ a_2^2\ C_1'\ (\alpha(1+\beta))^{|k'-k|} &, \text{ if } \ k^* < k, k' \le n \\ a_1 a_2\ C_1'\ (\alpha(1+\beta))^{|k'-k|} &, \quad \text{otherwise}, \end{cases}$$

where

$$C_1' = \frac{\beta(\lambda - 1)}{(1+\beta)(1-\alpha)(1-\bar{B})^2(1-\bar{B}^2\lambda)}$$

and $\bar{B} = \max\{B_1, B_2\}$.

In the following C stands for a positive constant whose value may change from line to line. Consistency of the estimator $\hat{\tau} = \hat{k}/n$ defined by (1.3) follows from the following theorem.

THEOREM 5.1 *Consider a sample* $r_1^2, \ldots, r_n^2$ *satisfying (5.1), (3.2) and the change–point estimator* $\hat{k}$ *given by (1.3). If Assumption 3.1 holds and*

$$\Delta := \frac{a_1}{1 - B_1} - \frac{a_2}{1 - B_2} \neq 0, \tag{5.2}$$

then for $\hat{\tau} = \hat{k}/n$

$$P\{|\hat{\tau} - \tau^*| > \epsilon\} \le \frac{C}{\epsilon^2 \Delta^2}\ n^{-1/2}, \tag{5.3}$$

where C is some positive constant.

PROOF: As a standard starting argument in the change-point problems, observe that $|ER_k|$ reaches its maximum at the point k^*. Indeed, by the equality $Er_{i,k}^2 = a_i/(1 - B_i)$, which is a necessary condition for the stationarity of the sequence $\{r_{i,k}^2\}$,

$$ER_k = \begin{cases} \Delta\ \tau(1 - \tau^*), \text{ if } & k \le k^* \\ \Delta\ \tau^*(1 - \tau), \text{ if } & k > k^*, \end{cases} \tag{5.4}$$

where $k = \tau n$. Thus

$$ER_{k^*} = \Delta\ \tau^*(1 - \tau^*). \tag{5.5}$$

It follows from (5.4), (5.5) that

$$|ER_{k^*}| - |ER_k| = \begin{cases} |\Delta|\ (\tau^* - \tau)(1-\tau^*), & \text{if} \quad k \le k^*, \\ |\Delta|\ (\tau - \tau^*)\tau^*, & \text{if} \quad k > k^* \end{cases}$$

and hence

$$|ER_{k^*}| - |ER_k| \ge \ |\Delta||\tau^* - \tau|(\tau^* \wedge (1-\tau^*)). \tag{5.6}$$

Observe now that

$$|R_k| - |R_{k^*}| \le |R_k - ER_k| + |ER_k| + |R_{k^*} - ER_{k^*}| - |ER_{k^*}|$$

$$\le 2 \max_{1\le k\le n} |R_k - ER_k| + |ER_k| - |ER_{k^*}|$$

and thus by (5.6)

$$|\Delta|\ |\tau^* - \tau|(\tau^* \wedge (1-\tau^*)) \le |ER_{k^*}| - |ER_k| \tag{5.7}$$

$$\le 2 \max_{1\le k\le n} |R_k - ER_k| + |R_{k^*}| - |R_k|.$$

Replacing in (5.7) τ by $\hat\tau$ and noting that $|R_{k^*}| \le |R_{\hat k}|$, we get

$$|\Delta|\ |\tau^* - \hat\tau|(\tau^* \wedge (1-\tau^*)) \le 2 \max_{1\le k\le n} |R_k - ER_k|. \tag{5.8}$$

It now remains to estimate $\max_{1\le k\le n} |R_k - ER_k|$. Observing that

$$|R_k - ER_k| = \frac{1}{n^2} |n \sum_{j=1}^{k} (r_j^2 - Er_j^2) - k \sum_{j=1}^{n} (r_j^2 - Er_j^2)|$$

$$\le \frac{1}{n} |\sum_{j=1}^{k} (r_j^2 - Er_j^2)| + \frac{1}{n} |\sum_{j=1}^{n} (r_j^2 - Er_j^2)|,$$

we get

$$\max_{1\le k\le n} |R_k - ER_k| \le 2 \max_{1\le k\le n} \frac{1}{n} |\sum_{j=1}^{k} (r_j^2 - Er_j^2)|. \tag{5.9}$$

Applying Theorem 4.1 with $m=1$, $X_k = r_k^2 - Er_k^2$ and $c_1 = \ldots = c_n = 1/n$ we obtain

$$\epsilon^2 P\{\max_{1\le k\le n} \frac{1}{n} |\sum_{j=1}^{k} (r_j^2 - Er_j^2)| > \epsilon\} \tag{5.10}$$

$$\le \frac{2}{n^2} \sum_{k=1}^{n-1} E\Big(|r_{k+1}^2 - Er_{k+1}^2|\ |\sum_{j=1}^{k} (r_j^2 - Er_j^2)|\Big) + \frac{1}{n^2} \sum_{k=0}^{n-1} E(r_{k+1}^2 - Er_{k+1}^2)^2.$$

$$\le \frac{2}{n^2} \sum_{k=1}^{n-1} (\mathrm{Var}\, r_{k+1}^2)^{1/2} \Big(\sum_{j,j'=1}^{k} \mathrm{Cov}(r_j^2, r_{j'}^2)\Big)^{1/2} + \frac{1}{n^2} \sum_{k=0}^{n-1} \mathrm{Var}\, r_{k+1}^2.$$

By Corollary 5.1

$$\mathrm{Cov}(r_j^2, r_{j'}^2) \le \Lambda((1+\beta)\alpha)^{|j-j'|},$$

where $\Lambda = C_1' \max\{a_1^2, a_2^2, a_1 a_2\}$. Denoting $\max\{\operatorname{Var} r_{1,0}^2, \operatorname{Var} r_{2,0}^2\} =: S^2$, it follows that

$$\epsilon^2 P\{\max_{1\le k\le n} \frac{1}{n} |\sum_{j=1}^{k} (r_j^2 - E r_j^2)| > \epsilon\} \tag{5.11}$$

$$\le \frac{2S\sqrt{\Lambda}}{n^2} \sum_{k=1}^{n-1} \left(\sum_{j,j'=1}^{k} ((1+\beta)\alpha)^{|j-j'|} \right)^{1/2} + \frac{1}{n}\, S^2$$

$$\le \frac{K_1}{n^2} \sum_{k=1}^{n-1} k^{1/2} + \frac{1}{n}\, S^2 \le K_2 n^{-1/2},$$

where K_1 and K_2 are positive constants.

Combining (5.8), (5.9) and (5.11) yields (5.3).

Consider now the change–point model (3.25) where the ARCH(1) sequences start from $k = 0$ rather than from $-\infty$.

THEOREM 5.2 *Consider the sample $r_1^2, \ldots, r_n^2$ satisfying (5.1), (3.25) and the change–point estimator $\hat{k}$ defined by (1.3). Then for $\hat{\tau} = \hat{k}/n$*

$$P\{|\hat{\tau} - \tau^*| > \epsilon\} \le \frac{C}{\epsilon^2 \Delta^2}\, n^{-1/2}. \tag{5.12}$$

PROOF: Denote $b := \max\{b_1, b_2\}$. Using Theorem 3.2, we obtain similarly as in the proof of Theorem 5.1

$$\epsilon^2 P\{\max_{1\le k\le n} \frac{1}{n} |\sum_{j=1}^{k} (r_j^2 - E r_j^2)| > \epsilon\} \tag{5.13}$$

$$\le \frac{2S}{n^2} \sum_{k=1}^{n-1} \left(\sum_{j,j'=1}^{k} (U\, b^{|j-j'|} + |V|\, b^{j+j'} \lambda^{j\wedge j'}) \right)^{1/2} + \frac{1}{n}\, S^2,$$

where U and V are defined in (3.27). The summands under the square root can be estimated as follows

$$\sum_{j,j'=1}^{k} (U\, b^{|j-j'|} + |V|\, b^{j+j'} \lambda^{j\wedge j'}) \le U \sum_{|j|<k} (k - |j|) b^{|j|} + |V| \sum_{j=1}^{k} (b^2\lambda)^j + 2|V| \sum_{j=1}^{k} \sum_{j'=1}^{j} (b^2\lambda)^j$$

$$\le Uk \sum_{|j|<k} b^{|j|} + |V| \frac{b^2\lambda}{1 - b^2\lambda} + 2|V| k\, \frac{b^2\lambda}{1 - b^2\lambda} \le K_3 k.$$

This, together with (5.13) and (5.8), yields (5.12). □

Using inequality (5.8) it is easy to verify that the estimator $\hat{\tau}$ is still consistent when the difference $\Delta = \Delta(n)$ defined by (5.2) tends to zero, as $n \to \infty$, at a rate slower that $n^{-1/4}$. Namely, suppose that

$$r_k^2 = (r_k^{(n)})^2 = \begin{cases} (r_{1,k}^{(n)})^2\ , & \text{if} \quad 1 \le k \le k^*, \\ (r_{2,k}^{(n)})^2\ , & \text{if} \quad k^* < k \le n, \end{cases} \tag{5.14}$$

where $k^* = [n\tau^*]$, $\tau^* \in (0,1)$ and $(r_{i,k}^{(n)})^2$, $i = 1,2$, are given by equations

$$(r_{i,k}^{(n)})^2 = (\sigma_{i,k}^{(n)})^2 \varepsilon_k^2, \quad (\sigma_{i,k}^{(n)})^2 = a_i^{(n)} + \sum_{j=1}^{\infty} b_i^{(n)}(j)(r_{i,k-j}^{(n)})^2. \tag{5.15}$$

Denote $B_i^{(n)} := b_i^{(n)}(1) + b_i^{(n)}(2) + \ldots$

Then we have the following corollary.

COROLLARY 5.2 *Let the sample $r_1^2, \ldots, r_n^2$ be given by (5.14) and (5.15) and Assumption 3.1 (with fixed α and β) for any $n \geq 1$ be satisfied. Assume that $a_1^{(n)}$, $a_2^{(n)}$ are bounded sequences and*

$$\Delta(n) := a_2^{(n)}/(1 - B_2^{(n)}) - a_1^{(n)}/(1 - B_1^{(n)})$$

satisfies $\Delta(n) \to 0$ and $\Delta^4(n)n \to \infty$ as $n \to \infty$. Then $\hat{\tau}$ is a weakly consistent estimator of τ^*.

The same result holds in the case of ARCH(1) sequences (3.25).

6 Rate of convergence

In Section 5 the consistency of the estimator was established. Using this result, we will show in the present section that $|\hat{\tau} - \tau^*| = O_P(\frac{1}{n})$, i.e.

$$\lim_{M\to\infty} \sup_n P\{|\hat{\tau} - \tau^*| > M/n\} = 0. \tag{6.1}$$

Relation (6.1) shows that the estimator $\hat{\tau}$ converges at the same rate as the least squares estimator studied by Bai (1994) in the case of weakly dependent linear sequences. The general idea of our proof is similar to Bai's proof, but given a different form of the Hájek-Rényi inequality, there are some differences. We have also been able to avoid a number of technical arguments used by Bai.

THEOREM 6.1 *Let the r_k^2 be given by (5.1), (3.2) and let the change–point estimator $\hat{k}$ be given by (1.3). If Assumption 3.1 holds for any n, $\Delta \neq 0$ is fixed then*

$$|\hat{\tau} - \tau^*| = O_P(\frac{1}{n}).$$

(Recall that $\tau^* \in (0,1)$ is fixed and $k^* = [n\tau^*]$.)

PROOF: Observe that

$$\begin{aligned}
\{|\hat{\tau} - \tau^*| > M/n\} &\subset \{\hat{k} - k^* > M\} \cup \{\hat{k} - k^* < -(M-1)\} \\
&\subset \{\max_{k \leq k^*+M} |R_k| < \max_{1\leq k\leq n} |R_k|\} \cup \{\max_{k\leq k^*-(M-1)} |R_k| = \max_{1\leq k\leq n} |R_k|\} \\
&\subset \{\max_{k \geq k^*+M} |R_k| = \max_{1\leq k\leq n} |R_k|\} \cup \{\max_{k\leq k^*-(M-1)} |R_k| = \max_{1\leq k\leq n} |R_k|\} \\
&\subset \{\max_{k \geq k^*+M} |R_k| \geq |R_{k^*}|\} \cup \{\max_{k\leq k^*-(M-1)} |R_k| \geq |R_{k^*}|\}.
\end{aligned}$$

Thus

$$P\{|\hat{\tau}-\tau^*|>M/n\} \le P\{\max_{k\ge k^*+M}|R_k|\ge|R_{k^*}|\}+P\{\max_{k\le k^*-(M-1)}|R_k|\ge|R_{k^*}|\}.$$

Because of symmetry we focus only on the first term. Since $\{|x|\ge|y|\}=\{x-y\ge 0, x+y\ge 0\}\cup\{x-y\le 0, x+y\le 0\}$, we have

$$P\{\max_{k\ge k^*+M}|R_k|\ge|R_{k^*}|\}$$

$$\le P\{\max_{k\ge k^*+M}R_k-R_{k^*}\ge 0\}+P\{\min_{k\ge k^*+M}R_k+R_{k^*}\le 0\}=: i_1+i_2.$$

We will show that $\lim_{M\to\infty}\sup_n i_1=0$ and $\lim_{M\to\infty}\sup_n i_2=0$. Without loss of generality assume that $\Delta>0$, so that $ER_k\ge 0$ (see (5.4)).

We consider only i_1. The term i_2 is dealt with analogously. We have

$$\{R_k-R_{k^*}\ge 0\}=\{R_k-ER_k-R_{k^*}+ER_{k^*}\ge ER_{k^*}-ER_k\}.$$

Observe that

$$R_k-ER_k-R_{k^*}+ER_{k^*}=n^{-2}\left(n\sum_{j=k^*+1}^{k}(r_j^2-Er_j^2)-(k-k^*)\sum_{j=1}^{n}(r_j^2-Er_j^2)\right).$$

Thus

$$\{R_k-R_{k^*}\ge 0\}\subset\{n^{-1}|\sum_{j=k^*+1}^{k}(r_j^2-Er_j^2)|+n^{-2}(k-k^*)|\sum_{j=1}^{n}(r_j^2-Er_j^2))|\ge ER_{k^*}-ER_k\}.$$

We know from (5.6) (recall that we assume $\Delta>0$ so that $ER_{k^*}\ge 0$, $ER_k\ge 0$) that for some $C>0$

$$ER_{k^*}-ER_k\ge C\Delta\ \frac{|k^*-k|}{n},$$

so that

$$i_1=P\{\max_{k\ge k^*+M}R_k-R_{k^*}\ge 0\}$$

$$\le P\{\max_{k\ge k^*+M}(k-k^*)^{-1}|\sum_{j=k^*+1}^{k}(r_j^2-Er_j^2)|\ge C\Delta/2\}+P\{n^{-1}|\sum_{j=1}^{n}(r_j^2-Er_j^2))|\ge C\Delta/2\}=: j_1+j_2.$$

Obviously

$$P\{n^{-1}|\sum_{j=1}^{n}(r_j^2-Er_j^2)|\ge C\Delta/2\}\le\frac{\tilde{C}}{\Delta^2}\ n^{-1/2}$$

for some $\tilde{C}>0$, so that $j_2\to 0$ as $n\to\infty$.

To prove that $j_1\to 0$ we use Theorem 3.1 and Theorem 4.1 with $c_k=1/k$ and $X_j=r_{j+p}^2-Er_{j+p}^2$ where p is an arbitrary integer. The third term on the right-hand side of (4.3) is then of order $m^{-1/2}$, whereas the remaining terms are of the order m^{-1}. So we obtain

$$P\{\max_{m\le k\le n}\frac{1}{k}|\sum_{j=1}^{k}(r_{j+p}^2-Er_{j+p}^2)|>\epsilon\}\le K\epsilon^{-2}m^{-1/2} \tag{6.2}$$

for some $K > 0$. Thus by (6.2)

$$j_1 \leq P\{\max_{l \geq M} \frac{1}{l} |\sum_{i=1}^{l} (r_{i+k^*}^2 - Er_{i+k^*}^2)| \geq C\Delta/2\} \leq \frac{4K}{C^2\Delta^2} M^{-1/2} \to 0$$

as $M \to \infty$. □

REMARK 6.1 Replacing the inequality $k \geq k^* + M$ by $k \geq k^* + M/\Delta^4$ in the proof of Theorem 6.1, it can be easily verified that

$$|\hat{\tau} - \tau^*| = O_P\left(\frac{1}{n\Delta^4}\right),$$

where $\Delta = \Delta(n)$ also depends on n.

7 Simulations and a data example

In this section we briefly discuss the sampling distribution of the change–point estimator $\hat{k}$ and then apply it to two financial time series.

The estimator $\hat{k}$ shares most properties of analogous CUSUM type estimators, see e.g. Antoch *et al.* (1997) and Horváth and Kokoszka (1997). It is skewed to the left, and the degree of skewness increases as the change–point k^* moves to the right. Estimates are most accurate if k^* lies around the middle of the sample. Obviously, the accuracy increases with the sample size and the size of the difference of variancesbefore and after k^*. Interestingly, it appears that the estimator $\hat{k}$ has larger spread when the change in variance is due to the change in the $b(j)$ rather than a. All these properties are illustrated by the histograms below. Each histogram describes the sample distribution of one thousand replications of $\hat{k}$ for several ARCH(1) models. Series of length $n = 10^3$ were estimated. In Figures 2 and 4 $k^* = 500$, in Figure 3 $k^* = 750$. In all Figures $a_1 = 1$ and $b(1) = 0.1$. In the left panels the variance increases by 30%, in the right panels it increases by 90%.

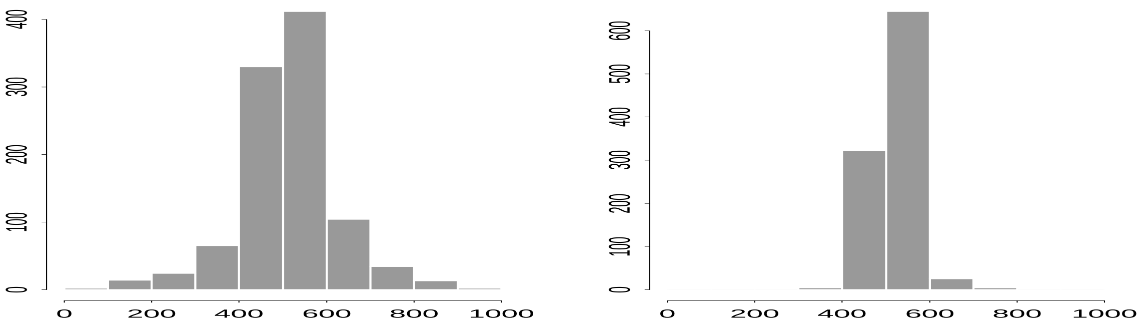

FIG. 2. $b(1) = b(2) = 0.1$. Left panel: $a_2 = 1.3$. Right panel: $a_2 = 1.8$.

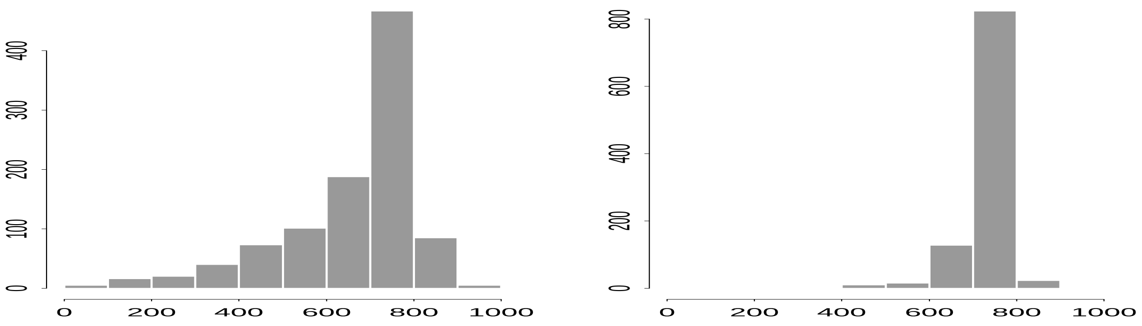

FIG. 3. $b(1) = b(2) = 0.1$. Left panel: $a_2 = 1.3$. Right panel: $a_2 = 1.8$.

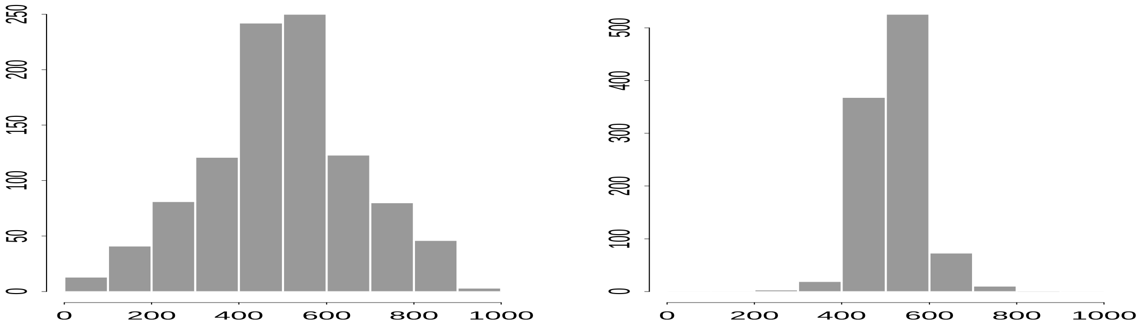

FIG. 4. Left panel: $a_2 = 1.25, b(2) = .013$. Right panel: $a_2 = 1.6, b(2) = .02$.

We conclude this section by applying our method to series of returns on foreign exchange rates. If P_t is the price of a currency in US dollars on day t, the return is defined as $r_t = \log P_t - \log P_{t-1}$. The left panel of Figure 5 shows 6630 daily returns on DM exchange rate from 24 August 1971 to 21 January 1997. Recall that 24 August 1971 marks the end of the Bretton Woods fixed exchange rates system. The right panel shows the sequences R_k for returns on DM and Pound exchange rates. The peak around $k = 2100$ corresponds to 1979, when the European Monetary System (EMS) was introduced. Interestingly, even though the United Kingdom joined the EMS in October 1990 and left two years afterwards, the impact of the system on the volatility of the Pound is similar as on the DM.

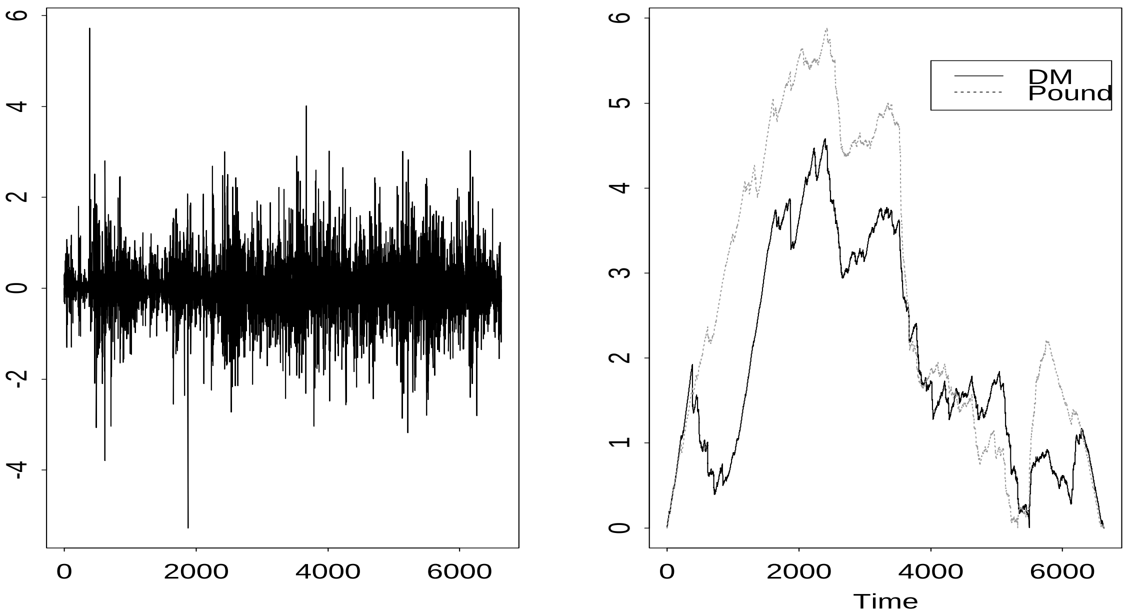

FIG. 5. Analysis of returns on exchange rates.

8 Appendix

PROOF OF THEOREM 3.2. Applying recursively equations (3.25), we obtain for $i = 1, 2$

$$r_{i,k}^2 = a_i \sum_{j=0}^{k-1} b_i^j \prod_{m=0}^{j} \varepsilon_{k-m}^2 + b_i^k r_{i,0}^2 \prod_{m=0}^{k-1} \varepsilon_{k-m}^2. \tag{8.1}$$

Without loss of generality, suppose that $k \le k'$. Then, multiplying $r_{1,k}^2$ and $r_{2,k'}^2$, due to (8.1), we can write $Er_{1,k}^2 r_{2,k'}^2$ as follows:

$$Er_{1,k}^2 r_{2,k'}^2 = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 := a_1 a_2 \sum_{j=0}^{k-1} \sum_{j'=0}^{k'-1} b_1^j b_2^{j'} E \prod_{m=0}^{j} \varepsilon_{k-m}^2 \prod_{m'=0}^{j'} \varepsilon_{k'-m'}^2,$$

$$I_2 := a_1 b_2^{k'} Er_{2,0}^2 \sum_{j=0}^{k-1} b_1^j E \prod_{m=0}^{j} \varepsilon_{k-m}^2 \prod_{m'=0}^{k'-1} \varepsilon_{k'-m'}^2,$$

$$I_3 := a_2 b_1^k Er_{1,0}^2 \sum_{j=0}^{k'-1} b_2^{j'} E \prod_{m'=0}^{j'} \varepsilon_{k'-m'}^2 \prod_{m=0}^{k-1} \varepsilon_{k-m}^2,$$

$$I_4 := b_1^k b_2^{k'} E(r_{1,0}^2 r_{2,0}^2) E \prod_{m=0}^{k-1} \varepsilon_{k-m}^2 \prod_{m'=0}^{k'-1} \varepsilon_{k'-m'}^2.$$

Term I_1 can be split into two terms, where first of them has two independent products:

$$I_1 := a_1 a_2 \sum_{j=0}^{k-1} \sum_{j'=0}^{k'-k-1} b_1^j b_2^{j'} E \prod_{m=0}^{j} \varepsilon_{k-m}^2 \prod_{m'=0}^{j'} \varepsilon_{k'-m'}^2$$

$$+ a_1 a_2 \sum_{j=0}^{k-1} \sum_{j'=k'-k}^{k'-1} b_1^j b_2^{j'} E \prod_{m=0}^{j} \varepsilon_{k-m}^2 \prod_{m'=0}^{j'} \varepsilon_{k'-m'}^2 =: I_{11} + I_{12}.$$

Here

$$I_{11} = a_1 a_2 \sum_{j=0}^{k-1} \sum_{j'=0}^{k'-k-1} b_1^j b_2^{j'} = a_1 a_2 \frac{(1-b_1^k)(1-b_2^{k'-k})}{(1-b_1)(1-b_2)}$$

$$= c_1 - c_1 b_1^k + c_1 b_1^k b_2^{k'-k} - c_1 b_2^{k'-k},$$

where

$$c_1 = \frac{a_1 a_2}{(1-b_1)(1-b_2)}.$$

Suppose that $b_1\lambda \neq 1$ and $b_2\lambda \neq 1$. The term I_{12}, after straightforward algebra, can be written as

$$I_{12} = a_1 a_2 \lambda \sum_{j=1}^{k} b_2^{k'-j} \left(1 + b_1\lambda + \ldots + (b_1\lambda)^{k-j-1} + (b_1\lambda)^{k-j}(1 + b_1 + \ldots + b_1^{j-1})\right)$$

$$= a_1 a_2 \lambda \sum_{j=1}^{k} b_2^{k'-j} \left(\frac{1-(b_1\lambda)^{k-j}}{1-b_1\lambda} + (b_1\lambda)^{k-j} \frac{1-b_1^j}{1-b_1}\right)$$

$$= c_2 b_2^{k'-k} + c_3 (b_1\lambda)^k b_2^{k'} + c_4 b_2^{k'} + c_5 b_1^k b_2^{k'-k},$$

where the constants c_2, c_3, c_4 and c_5 are

$$c_2 = a_1 a_2 \left(\frac{\lambda}{(1-b_2)(1-b_1\lambda)} - \frac{\lambda}{(1-b_1\lambda)(1-b_1b_2\lambda)} + \frac{\lambda}{(1-b_1)(1-b_1b_2\lambda)}\right),$$

$$c_3 = a_1 a_2 \left(\frac{\lambda}{(1-b_1\lambda)(1-b_1b_2\lambda)} - \frac{\lambda}{(1-b_1)(1-b_1b_2\lambda)} + \frac{\lambda}{(1-b_1)(1-b_2\lambda)}\right),$$

$$c_4 = -a_1 a_2 \frac{\lambda}{(1-b_2)(1-b_1\lambda)},$$

$$c_5 = -a_1 a_2 \frac{\lambda}{(1-b_1)(1-b_2\lambda)},$$

For the term I_2 we have:

$$I_2 = a_1 b_2^{k'} E r_{2,0}^2 \sum_{j=0}^{k-1} b_1^j \lambda^{j+1} = -c_4 b_2^{k'} + c_4 (b_1\lambda)^k b_2^{k'}.$$

Similarly as in the case of the term I_{12} we obtain that

$$I_3 = a_2 b_1^k E r_{1,0}^2 \left(1 + b_2 + \ldots + b_2^{k'-k-1} + b_2^{k'-k}\lambda(1 + b_2\lambda + \ldots + (b_2\lambda)^{k-1})\right)$$

$$= a_2 b_1^k E r_{1,0}^2 \left(\frac{1-b_2^{k'-k}}{1-b_2} + b_2^{k'-k}\lambda \frac{1-(b_2\lambda)^k}{1-b_2\lambda}\right)$$

$$= c_1 b_1^k + c_6 b_1^k b_2^{k'-k} + c_5 (b_1\lambda)^k b_2^{k'},$$

where

$$c_6 = a_1 a_2 \left(-\frac{1}{(1-b_1)(1-b_2)} + \frac{\lambda}{(1-b_1)(1-b_2\lambda)} \right) = -c_1 - c_5.$$

Finally

$$I_4 = E(r_{1,0}^2 r_{2,0}^2)(b_1\lambda)^k b_2^{k'}.$$

Gathering the corresponding coefficients in the sum $I_{11} + I_{12} + I_2 + I_3 + I_4$, we see that

$$E(r_{1,k}^2 r_{2,k'}^2) = \frac{a_1 a_2}{(1-b_1)(1-b_2)} + (-c_1 + c_1) b_1^k + (c_4 - c_4) b_2^{k'} + (c_1 + c_5 + c_6) b_1^k b_2^{k'-k}$$

$$+(-c_1 + c_2) b_2^{k'-k} + (c_3 + c_4 + c_5)(b_1\lambda)^k b_2^{k'} = \frac{a_1 a_2}{(1-b_1)(1-b_2)} + U b_2^{k'-k} + V (b_1\lambda)^k b_2^{k'}.$$

Applying now equality

$$E r_{1,k}^2 E r_{2,k'}^2 = \frac{a_1 a_2}{(1-b_1)(1-b_2)},$$

we obtain the statement of theorem in the case where $b_1\lambda \neq 1$ and $b_2\lambda \neq 1$. The claim in the case $b_1\lambda = 1$ or $b_2\lambda = 1$ can be verified easily. □

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