Forecasting of density functions with an application to cross-sectional and intraday returns

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\begin{abstract}
This paper is concerned with the forecasting of probability density functions. Density functions are nonnegative and have a constrained integral, and thus do not constitute a vector space. The implementation of established functional time series forecasting methods for such nonlinear data is therefore problematic. Two new methods are developed and compared to two existing methods. The comparison is based on the densities derived from cross-sectional and intraday returns. For such data, one of our new approaches is shown to dominate the existing methods, while the other is comparable to one of the existing approaches.
\end{abstract}

\section{Introduction}

There are many problems in which the prediction of future probability densities functions is useful; for example, the prediction of densities of income, fertility, mortality, and densities of several types of returns on financial assets. Motivated by the need to forecast cross-sectional and intraday returns, we propose two new methods of predicting densities, and compare them to two existing methods.

The importance of density forecasting in finance has long been recognized, and is stated aptly by Crnkovic and Drachman (1997, p. 47) as follows: “At the heart of market risk measurement is the forecast of the probability density functions (PDFs) of the relevant market variables ... a forecast of a PDF is the central input into any decision model for asset allocation and/or hedging ... therefore, the quality of risk management will be considered synonymous with the quality of PDF forecasts”. Also, as was pointed out by Ross (2017), forecasting the density of financial returns allows us to recover the pricing kernel, the market risk premium and the probability of a catastrophe, and to construct model-free tests of the efficient market hypothesis. The recovery theorem of Ross (2017) enables one to determine the market’s forecast of returns. In financial risk management, there is more interest in the density of the future returns, as better density forecasts can produce better risk forecasts in terms of the value-at-risk (VaR) or conditional-value-at-risk (CVaR) forecast. Lee, Xi, and Zhang (2014) proposed a multiplicative decomposition of the financial returns in order to improve the density forecasts of financial returns and show that the risk forecasts produced from the density forecast using the decomposition and maximum entropy are superior to the approaches used more broadly, especially in extreme tail events of a large loss. This paper enriches the set of tools that can be used for forecasting the densities that are relevant in financial applications. Examples of the most commonly studied densities are shown in Fig. 1.

While density functions, or alternative characterizations of their underlying distributions, may be thought of as elements of a Hilbert space, they do not constitute a linear subspace. If the densities are treated as elements of $L^2$, the predicted curves will be elements of $L^2$, but will not
necessarily be densities. This is why the usual methods of prediction, which are based on Hilbert space formalism and go back to the work of Kolmogorov and Wiener, are not applicable directly. A natural way of dealing with such constraints is to peel them away by means of an invertible transformation that maps densities onto a linear space. Invertibility is key, as it allows one to perform analyses in the transformed space, then visualize and interpret the results in terms of densities by means of the inverse map. We propose two approaches that fall into this general paradigm: one based on compositional data analysis, the other on a suitable transformation of densities studied by Petersen and Müller (2016). Recent prediction methods for unconstrained curves are explained by Kokoszka and Reimherr (2017, Chapter 8), see also Hyndman and Ullah (2007) and Shang and Hyndman (2011). The forecasting of functional time series with constraints that differ from those considered here has been considered by Canale and Vantini (2016), for example.

Fig. 1. Densities of cross-sectional monthly returns.

The paper is organized as follows. Section 2 describes the approaches that we study, with a focus on the two new approaches (Sections 2.1 and 2.2). Section 3 is dedicated to the comparison of the methods, with its last subsection summarizing the findings of this research.

2. Forecasting approaches

This section describes the forecasting approaches that we consider in this paper. The approaches based on compositional data analysis (Section 2.1) and the log quantile transformation (Section 2.2) are new. The approach introduced in Section 2.3 was proposed by Horta and Ziegelmann (2018) for forecasting the densities of intra-day returns. The three approaches above observe the densities \( f_t, t = 1, 2, \ldots, n \), and we want to forecast future densities \( f_{n+h}, h \geq 1 \). The parametric approach presented in Section 2.4, based on fitting skewed \( t \)-distributions, was proposed by Wang (2012) in the context of predicting monthly cross-sectional returns. It uses past returns rather than past densities as inputs.

2.1. Compositional data analysis

Density functions, which are nonnegative functions that integrate to one, share some features with compositional data (Aitchison, 1986; Pawlowsky-Glahn, Egozcue, & Tolosana-Delgado, 2015). Compositional data are defined as a random vector of \( K \) nonnegative components, \( \mathbf{D} = (d_1, \ldots, d_K) \), the sum of which is a specified constant, typically set equal to 1 (portion), 100 (percentage) or \( 10^6 \) (parts per million). Thus, sample space of compositional data is the simplex

\[
S^K = \left\{ \mathbf{D} = (d_1, \ldots, d_K)^T, \quad d_k > 0, \quad \sum_{k=1}^{K} d_k = c \right\},
\]

where \( c \) is a fixed constant, and \( ^T \) denotes the transpose. The simplex sample space is a \( K - 1 \) dimensional subset of the Euclidean space \( \mathbb{R}^K \).

Compositional data arise in many scientific fields, such as geology (geochemical elements), economics (income/
expenditure distribution), medicine (body composition), food industry (food composition), chemistry (chemical composition), agriculture (nutrient balance biomics), environmental sciences (soil contamination), ecology (abundance of different species), and demography (life-table death counts). For example, Scealy, de Caritat, Grunsky, Tsagris, and Welsh (2015) study the concentration of chemical elements in sediment and rock samples, while (Scealy & Welsch, 2014) analyze households’ total weekly expenditure on food and housing costs. In the field of demography, Boucher, Canudas-Romo, Oeppen, and Vaupel (2017) treat the age-specific life-table death count \(d_k\) as compositional data and use a principal component approach to forecast life-table death counts.

The method that we propose consists of the following steps.

1. Compute the geometric mean function

\[
\alpha_n(u) = \exp \left\{ \frac{1}{n} \sum_{t=1}^{n} \ln[f_t(u)] \right\}.
\]

and set

\[
s_t(u) = f_t(u)/\alpha_n(u) \int f_t(u)/\alpha_n(u) du.
\]

The geometric mean standardizes the ranges, so that no range dominates the weighting. The standardization in Eq. (2) ensures that \(\int s_t(u) du = 1\), a condition that is imposed commonly in compositional data analysis.

2. Apply the centered log-ratio transformation given by

\[
\beta_t(u) = \ln \left( \frac{s_t(u)}{g_t} \right),
\]

where \(g_t\) is the geometric mean given by

\[
g_t = \exp \left\{ \int \ln[s_t(u)] du \right\}.
\]

The log-ratio transformation in Eq. (3) removes the constraints on \(f_t\).

3. Apply FPC analysis to the transformed data \(\{\beta_1(u), \ldots, \beta_n(u)\}\), i.e., compute the Karhunen–Loève expansions

\[
\beta_t(u) = \sum_{\ell=1}^{n} \hat{\beta}_{t,\ell} \hat{\phi}_\ell(u) \approx \sum_{\ell=1}^{L} \hat{\beta}_{t,\ell} \hat{\phi}_\ell(u),
\]

where \(\hat{\phi}_1(u), \ldots, \hat{\phi}_n(u)\) are the first \(L\) estimated FPCs, and \(\hat{\beta}_{1,1}, \ldots, \hat{\beta}_{L,1}\) are their scores.

We determine \(L\) by the explained variance criterion:

\[
L = \arg\min_{L \geq 1} \left\{ \sum_{\ell=1}^{L} \hat{\lambda}_\ell \left/ \sum_{\ell=1}^{n} \hat{\lambda}_\ell \right. \geq \delta \right\},
\]

where \(\hat{\lambda}_t\) is the sample variance of the \(\hat{\beta}_{t,\ell}\), \(t = 1, 2, \ldots, n\). In our implementation, we use \(\delta = 85\%\) (see e.g. Horváth & Kokoszka, 2012, p. 41).

4. Forecast the \(L\) FPC scores. This can be done in many ways. We use an exponential smoothing forecasting method. Alternatively, a univariate ARIMA method or a multivariate time series forecasting method (Aue, Norinho, & Hörmann, 2015) can be used.

Obtain the \(h\)-step-ahead forecast \(\hat{\beta}_{n+h,\ell}\) of the \(\ell\)th principal component score. We utilize an automatic algorithm developed by Hyndman and Kadakar (2008) to determine the optimal exponential smoothing model based on the corrected Akaike information criterion (AIC) of Hurvich and Tsai (1993).

Conditioning on the estimated principal components and the observed data, the forecast of \(\hat{\beta}_{n+h}(u)\) is given by

\[
\hat{\beta}_{n+h}(u) = \sum_{\ell=1}^{L} \hat{\beta}_{n+h,\ell} \hat{\phi}_\ell(u).
\]

5. Transform back to the compositional data; i.e., take the inverse centered log-ratio transformation given by

\[
\hat{s}_{n+h}(u) = \frac{\exp[\hat{\beta}_{n+h}(u)]}{\int \exp[\hat{\beta}_{n+h}(u)] du},
\]

where \(\hat{s}_{n+h}(u)\) denotes the forecasts in Eq. (5).

6. Finally, we add back the geometric means, to obtain the forecasts of the density function

\[
\hat{f}_{n+h}(u) = \frac{\hat{s}_{n+h}(u) \alpha_n(u)}{\int \hat{s}_{n+h}(u) \alpha_n(u) du},
\]

where \(\alpha_n(u)\) is the geometric mean function given in Eq. (1).

The functions \(\beta_t\) in Eq. (3) are given by

\[
\beta_t(u) = \ln f_t(u) - \frac{1}{n} \sum_{i=1}^{n} \ln[f_t(u)] - \int \frac{f_t(u)}{\alpha_n(u)} du.
\]

The middle term on the right-hand side does not depend on \(t\), and so has no impact on predictions. The third term is a normalizing constant that depends on \(t\), and could potentially either improve the predictions or make them worse. Following the advice of a referee, we explored how a modification without this standardization would perform. This is equivalent to working with \(\beta_t(u) = \ln(f_t(u))\), and using \(\hat{f}_{n+h}(u) = \exp \hat{\beta}_{n+h}(u)\) instead of steps 5 and 6. We will refer to this approach as CoDa (no standardization), and to the approach described in steps 1–6 as CoDa (standardization).

A related approach can be found in a recent paper by Hron, Menafoglio, Tempil, Hrůzová, and Filzmoser (2016), where the compositional data approach of Delicado (2011) was extended to define a version of functional principal component analysis on samples of densities. A related theoretical development is presented by Egozcue, Diaz-Barrero, and Pawlowsky-Glahn (2006). The method of Hron et al. (2016) cannot be applied to the densities we study, at least not without some fairly substantial modifications. Their setup is to begin with the collection of densities on an interval \([a, b]\), for which
the log density is square integrable. This key requirement means that the density must be positive on \([a, b]\) for their methodology to work. The support of return densities (i.e., the collection of grid points for which the estimated density is strictly positive) is different for each month or day, even within the same data set, and therefore the method of Hron et al. (2016) cannot be applied directly without first adjusting the estimates to be strictly positive on a common interval.

2.2. Log quantile density transformation

Our approach is based on the ideas of Petersen and Müller (2016). The general paradigm is as follows. We transform the densities \(f_t, \ t \leq n\), to the space \(L^2\) using a suitable transformation \(\psi\), and set \(Y_t = \psi(f_t)\). There are several prediction methods that are known to work well for densities in the “unrestricted” Hilbert space \(L^2\). Applying one of these methods, we obtain the predicted curves \(\hat{Y}_{t+h}, 1 \leq h \leq H\). We then apply the inverse transformation and obtain the predicted densities \(\hat{f}_{t+h} = \psi^{-1}(\hat{Y}_{t+h})\). The primary difficulty is that we must ensure that \(\psi^{-1}(\hat{Y}_{t+h})\) is a density. In the context of functional principal components analysis, Petersen and Müller (2016) considered two specific transformations: the log hazard transformation and the log quantile density (LQD) transformation. We explain in Appendix B why the log hazard transformation is not suitable for our purpose, and focus on the LQD transformation. The main theoretical novelty of our approach relative to that of Petersen and Müller (2016) is to define a modified LQD transformation that allows for densities with different supports to be transformed to functions on a common domain and analyzed jointly. Petersen and Müller (2016) studied the theoretical properties of FPCA after LQD transformation. The method has since been applied successfully in the context of functional regression models, where densities appear as predictors (Petersen, Chen, & Müller, 2019), as well as in a distributional regression framework in an engineering application (Chen, Bao, Li, & Spencer Jr, 2019). This paper is the first to demonstrate the utility of this approach for forecasting distributional time series. In particular, we are the first to provide algorithms for computing the modified LQD transformation and its inverse.

The density is only one of many characterizations of a distribution. Each representation has its own interpretation and properties, and we will demonstrate how these can be used to obtain a functional representation of the distribution that is free of nonlinear constraints. Let \(F_t\) and \(Q_t\) be the cumulative distribution function (cdf) and quantile functions corresponding to \(f_t\) respectively, i.e.

\[
F_t(x) = \int_{-\infty}^{x} f_t(u)du, \quad -\infty < x < \infty,
\]

\[
Q_t(s) = F_t^{-1}(s) = \inf\{y \in (0, 1) : F_t(y) \geq s\}, \quad s \in (0, 1).
\]

Each of the three functions, \(f_t\), \(F_t\) and \(Q_t\), uniquely characterizes the same distribution, and can be used as a functional data object. However, they are all subject to nonlinear constraints that make the application of typically linear functional data methods inadequate, namely that \(f_t \geq 0, \int_{-\infty}^{\infty} f_t(x)dx = 1, 0 \leq F_t(x) \leq 1, F_t' \geq 0\) and \(Q_t' \geq 0\), where \('\) denotes a first-order derivative. Of these three, \(Q_t\) is the least constrained, and it is easy to see that the so-called quantile density (Jones, 1992; Parzen, 1979; Tukey, 1965) \(q_t = Q_t'\) is only constrained to be nonnegative. Thus, all constraints can be removed completely by computing the log quantile density (LQD)

\[
Y_t(s) = \log(q_t(s)) = -\log(f_t \circ Q_t(s)), \quad s \in [0, 1].
\]

The last equality follows because

\[
q_t(s) := Q_t(s) = (F_t^{-1})'(s) = \frac{1}{F_t' \circ F_t^{-1}(s)} = \frac{1}{f_t \circ Q_t(s)}.
\]

However, the transformation from \(f_t\) to its LQD is not invertible, since for any constant \(c\), the quantile density of \(f_t(-c)\), the density shifted by \(c\) is also \(q_t\). Petersen and Müller (2016) assumed that the densities had a common known support \(T\), and developed a modification of the inverse transformation so that densities with support \(T\) could be mapped back and forth.

In our application, the densities do not always have the same support, so we are forced to deal with the non-invertibility issue in a different way. Although the supports differ in our data set, most have most of their mass concentrated around zero, and have bounded support with endpoints

\[
r^+_t = \inf \{r > 0 : f_t(r) = 0\}, \quad r^-_t = \sup \{r < 0 : f_t(r) = 0\}.
\]

Thus, in what follows, we assume that \(0 \in \text{supp}(f_t)\), which is the case for all densities that we consider. This will facilitate a modified definition of the log quantile density transformation that is indeed invertible, and thus suitable for our purposes. While the assumption \(0 \in \text{supp}(f_t)\) is natural for the data sets that we consider, this may be adapted to any sample of density functions for which the intersection of their supports contains a known point.

Definition A.1 in Appendix A defines a class \(\mathcal{D}\) of densities for which the steps described in the remainder of this section are justified. This section provides an algorithmic description. We first present two algorithms that show how the forward (density to \(L^2\)) and backward (\(L^2\) to density) transformations are executed. In fact, our transformations act between the space of densities and the product space \(L^2[0, 1] \times (0, 1)\), as is stated in the following definition.

Definition 2.1. For a density \(f \in \mathcal{D}\), with \(\mathcal{D}\) as in Definition A.1, let \(F\) and \(Q\) be the cdf and corresponding quantile function, and define

\[
Y(s) := -\log(f(Q(s))), \quad s \in [0, 1].
\]

The modified log quantile density transformation is the map

\[
\psi : \mathcal{D} \to L^2[0, 1] \times (0, 1), \quad \text{given by } \psi(f) = (Y, F(0)).
\]

Note that this definition includes the possibility that \(Y(0)\) and/or \(Y(1)\) is/are infinite. Proposition A.1 ensures
that $Y \in L^2[0, 1]$ and that the transformation $\psi$ is invertible. As was mentioned above, this definition can be generalized slightly under the assumption that there is a known point $x_0$ which lies in the support of all densities $f_t$. In this case, one would set $\psi(f) = (Y, F(x_0))$. An even more flexible alternative would be to choose a fixed percentile level $s_0 \in (0, 1)$ and define $\psi(f) = (Y, Q(s_0))$, which would not require any overlap of the supports of the $f_t$. However, for the sake of simplicity and to conform with the motivating data examples, we consider the specific form in Eq. (9).

Let $f \in \mathcal{D}$ be a density whose values are known on a discrete grid $r^{-} = x_0 < x_1 < \cdots < x_L = r^{+}$, where $x_j = 0$ for some $j$. In what follows, $c$ will represent the value $F(0)$, $s_l$ will represent grid points on $[0, 1]$, and $Y(s_l)$ is given by Eq. (8).

**Algorithm 1** Forward transformation

**Input:** Data pairs $(x_i, f(x_i))$, $l = 0, \ldots, L$

**Output:** Data pairs $(s_l, Y(s_l))$, $l = 0, \ldots, L$ and $c$

For $l = 1, \ldots, L$:
1. Compute $F(x_l) = \int_{x_0}^{x_l} f(x)dx$, $l = 1, \ldots, L$ by numerical integration of the pairs $(x_l, f(x_l))$.
2. Define $s_l = F(x_l)$ so that $Q(s_l) = x_l$.
3. Compute $Y(s_l) = -\ln(f(x_l)) = F(0)$.
4. Find $j$ such that $x_j = 0$ and set $c = F(x_j) = F(0)$.

For the inverse transformation, one begins with a function $g$ (corresponding to the output $Y$ of Algorithm 1) that is continuous on $[0, 1]$. Available are discrete observations $(s_l, g(s_l))$, where $0 = s_0 < s_1 < \cdots < s_L = 1$ is a grid. We also have a value $c \in (0, 1)$ that specifies the value of the cdf at zero for the target density $f$. We assume that $s_j = c$ for some $j \neq 0$ and $L$. Thus, $g$ is the predicted function, and $c$ corresponds to $\hat{F}_{n+h|n}(0)$.

**Algorithm 2** Backward transformation

**Input:** Data pairs $(s_l, g(s_l))$, $l = 0, \ldots, L$ and $c$

**Output:** Data pairs $(x_l, f(x_l))$, $l = 0, \ldots, L$

For $l = 1, \ldots, L$:
1. Find $j$ such that $s_j = c$.
2. Compute $Q(s_l) = \int_{s_0}^{s_l} \exp(g(s))ds$, $l = 1, \ldots, L$ by numerical integration of the pairs $(s_l, \exp(g(s_l)))$.
3. Define $x_l = Q(s_l)$ so that $f(x_l) = s_l$.
4. Compute $f(x_l) = \exp(-g(s_l))$.

Multiple LQD functions and/or densities can be computed using the above algorithms, and it is often desirable for these to be computed on a common grid. As an optional post-processing step to either algorithm, one can use linear interpolation to impute onto a common, equi-spaced grid of values.

With the above two algorithms, the full prediction procedure can be summarized in the following steps.

1. Transform the densities $f_t$ into $(Y_t, c_t)$, as in Algorithm 1, $t = 1, 2, \ldots, n$.
2. Compute the predictions $\hat{Y}_{n+h}, \hat{c}_{n+h}$, $h \geq 1$.
3. Transform each pair $(\hat{Y}_{n+h}, \hat{c}_{n+h})$ into the predicted density $\hat{f}_{n+h|n}$ as in Algorithm 2.

We predict $Y_{n+h} \in \mathcal{D}$ by first obtaining a mean estimate $\hat{\mu}_Y(s)$ and covariance estimate $\hat{C}_n$, from which we obtain eigenfunction estimates $\hat{v}_j$ and predicted FPC scores $\hat{\xi}_{n+h|n}$. The predicted value of the future LQD is then $\hat{Y}_{n+h|n} = \hat{\mu}_Y(s) + \sum_{j=1}^{p} \hat{\xi}_{n+h|n} \hat{v}_j(s)$, which would at least be continuous on $(0, 1)$, even if it diverges at the boundary. We use a scalar time series model to predict $c_{n+h} = F_{n+h}(0)$ using the time series of values $c_t = F_t(0)$, leading to a value $\hat{c}_{n+h|n}$. The prediction of the future density $\hat{f}_{n+h}$ is then given by $\hat{f}_{n+h|n}(\psi^{-1}(\hat{Y}_{n+h|n}), \hat{c}_{n+h|n})$, where $\psi^{-1}$ is the inverse transformation described in Algorithm 2.

### 2.3. Dynamic functional principal component regression

This method is based on the dynamic functional principal components (DFPCs) introduced by Bathia, Yao, and Ziegelmann (2010), which should not be confused with the spectral domain DFPCs developed by Hörmann, Kidziński, and Hallin (2015). The method is implemented exactly as described by Horta and Ziegelmann (2018), using their R software. We refer to their paper for its justification. It proceeds in the following steps.

1. Compute the kernel
\[
\hat{K}(u, v) = \frac{1}{(n-p)^2} \sum_{t=1}^{n-p} \sum_{k=1}^{p} \left( f_t(u) - \bar{f}(u) \right) \times \left( f_k(u) - \bar{f}(u) \right) \delta_{t+k-k} - \bar{f}
\]
and its $\hat{a}$ orthonormal eigenfunctions $\hat{\psi}_1(\cdot), \ldots, \hat{\psi}_a(\cdot)$, corresponding to nonzero eigenvalues.

2. Approximate the densities $f_t$, $t = 1, 2, \ldots, n$, by
\[
\hat{f}_t(u) = \bar{f}(u) + \sum_{j=1}^{\hat{a}} \hat{\eta}_j \hat{\psi}_j(u),
\]
\[
\hat{\eta}_j = \int \hat{f}_t(u) - \bar{f}(u) \hat{\psi}_j(u)du.
\]

3. Fit a vector autoregressive (VAR) model with the order selected by the AIC to the vectors
\[
\hat{\eta}_t = [\hat{\eta}_{t1}, \ldots, \hat{\eta}_a]^T,
\]
and compute the predicted vectors $\hat{\eta}_{n+h}$.

4. Compute the predictions
\[
\hat{f}_{n+h|n}(u) = \bar{f}(u) + \sum_{j=1}^{\hat{a}} \hat{\eta}_{n+h|n} \hat{\psi}_j(u).
\]

5. Since the predicted functions in the previous step do not have to be nonnegative and do not have to
integrate to one, set $f_{n+h|n}(u) = \max\{0, f_{n+h}(u)\}$ and
\begin{equation}
\hat{f}_{n+h|n}(u) = \frac{f_{n+h|n}(u)}{\int f_{n+h|n}(u)du}.
\end{equation}

2.4 Skewed $t$ distribution

As was noted above, this method has been used only in the context of the prediction of densities of cross-sectional returns. We follow exactly the approach of Wang (2012, Chapter 5), who also kindly provided the R code. Earlier work on modeling the cross-section of returns includes that by Lillo and Mantegna (2000) and Cont (2001). In particular, Cont (2001) emphasized that a parametric model must have at least four parameters in order to successfully reproduce the empirical properties: a location parameter, a scale parameter, a parameter that captures the decay in the tails, and an asymmetry parameter that allows the left and right tails to have different behaviors.

The skewed $t$ distribution has been studied extensively. For example, Jones and Faddy (2003) proposed a tractable form and the associated likelihood inference. Azzalini and Capitanio (2003) constructed a skewed $t$ distribution from a skewed normal distribution, while Fernández and Steel (1998) presented a general method for transforming any symmetric and unimodal distribution to a skewed distribution. Although these two methods are equivalent, we follow Fernández and Steel’s (1998) method for constructing a skewed $t$ distribution and fit it to the cross-sectional and intraday returns. The density function of this skewed $t$ distribution is
\begin{equation}
f(u|\mu, \sigma, \nu, \lambda) = \begin{cases} 
\sqrt{1-\lambda}(1+\lambda) & \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi\sigma^2\Gamma\left(\frac{\nu}{2}\right)}} 
\left(1 + \frac{(u-\mu)^2(1+\lambda)}{\lambda^2\nu(1-\lambda)}\right)^{-\frac{\nu+1}{2}}, \\
\sqrt{1-\lambda}(1+\lambda) & \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi\sigma^2\Gamma\left(\frac{\nu}{2}\right)}} 
\left(1 + \frac{(u-\mu)^2(1-\lambda)}{\lambda^2\nu(1+\lambda)}\right)^{-\frac{\nu+1}{2}},
\end{cases}
\end{equation}
where $-\infty < \mu < \infty$ is the location parameter, $\sigma > 0$ is the scale parameter, $-1 < \nu < 1$ is the skewness parameter, and $\nu > 0$ is degrees of freedom.

We estimate the four parameters by maximizing the logarithm of the log likelihood
\begin{equation}
\ln(L) = nC(\nu, \lambda) - \frac{\nu + 1}{2} \sum_{u_i > \mu} \ln\left(1 + \frac{(u_i - \mu)^2(1+\lambda)}{\lambda^2\nu(1+\lambda)}\right) - \frac{\nu + 1}{2} \sum_{u_i \leq \mu} \ln\left(1 + \frac{(u_i - \mu)^2(1+\lambda)}{\lambda^2\nu(1-\lambda)}\right),
\end{equation}
where
\begin{equation}
C(\nu, \lambda) = \ln \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi\sigma^2\Gamma\left(\frac{\nu}{2}\right)}} + \frac{1}{2} \ln \frac{(1-\lambda)(1+\lambda)}{\nu\pi\lambda^2}.
\end{equation}

The prediction method is summarized as follows.

1. For each $t = 1, 2, \ldots, n$, maximize the log likelihood in Eq. (11) to obtain the MLEs $\hat{\theta}_t = (\hat{\mu}_t, \hat{\sigma}_t, \hat{\nu}_t, \hat{\lambda}_t)$.

2. Transform the estimated parameters to obtain an approximately stationary vector-valued time series $\theta_t = [\theta_{t1}, \theta_{t2}, \theta_{t3}, \theta_{t4}]^T$, $t = 2, \ldots, n$,

where
\begin{equation}
\begin{align*}
\theta_{t1} &= \hat{\mu}_t, \\
\theta_{t2} &= \ln\left(\frac{\hat{\sigma}_t^2}{\hat{\sigma}_{t-1}^2}\right), \\
\theta_{t3} &= \ln\left(\frac{\hat{\nu}_t}{\hat{\nu}_{t-1}}\right), \\
\theta_{t4} &= \hat{\lambda}_t,
\end{align*}
\end{equation}

and where $\hat{\nu}_t$ is the cdf of the standard $t$ distribution with $\hat{\nu}_t$ degrees of freedom evaluated at 2; $\hat{\nu}_t = F_2(2)$. (The introduction of $\hat{\nu}_t$ smooths out the jumps in the values of $\hat{\nu}_t$.)

3. Fit a VAR model to the $\theta_t$, $t \leq n$, with the order selected via the AIC. Compute the forecasts $\hat{\theta}_{n+h}$, $h \geq 1$.

4. Using the components of $\hat{\theta}_{n+h}$, compute the predicted parameters $\hat{\mu}_{n+h|n}, \hat{\sigma}_{n+h|n}, \hat{\nu}_{n+h|n}, \hat{\lambda}_{n+h|n}$ and the predicted densities
\begin{equation}
\hat{f}_{n+h|n}(u) = f\left(u|\hat{\mu}_{n+h|n}, \hat{\sigma}_{n+h|n}, \hat{\nu}_{n+h|n}, \hat{\lambda}_{n+h|n}\right).
\end{equation}

3. Comparison of the four prediction methods

3.1. Data sets

We consider four financial data sets, two consisting of cross-sectional returns and two of five-minute intraday returns. Both are amongst the most extensively studied forms of financial data, with hundreds of contributions to date; see Harvey, Liu, and Zhu (2016) for a recent review of cross-sectional returns and Renault (2017) for intraday returns.

3.1.1. Dow Jones cross-sectional returns

The Dow Jones Industrial Average (DJIA) is a stock market index that shows how 30 large publicly-owned companies based in the United States have traded during a standard NYSE trading session. We consider monthly cross-sectional returns from April 2004 to December 2017. The data were obtained from the CRSP database (Center for Research in Security Prices). We thus have a sample of log-price observations, denoted by $(q_{it}, j = 1, \ldots, 30)$, for each month $t = 1, \ldots, n - 1$. We define the $j$th return in month $t$ as
\begin{equation}
r^*_{j,t} := q_{t+1,j} - q_{j,t}, \quad j = 1, 2, \ldots, 30,
\end{equation}
where $r^*_{j,t}$ is the log return for the $j$th company at the close of month $t$.

We can then estimate the density $f_t$ for each month $t$ from these data, as is explained in Section 3.2. Fig. 2 presents three density functions for three representative months that have mostly negative, mostly positive and close to zero returns, respectively.
3.1.2. S&P 500 cross-sectional returns

The Standard & Poor’s 500 (SPX) index is an American stock market index based on the market capitalizations of 500 large companies with common stocks listed on the NYSE or NASDAQ. We consider monthly cross-sectional returns from April 2004 to December 2017. These data were obtained from the CRSP database.

3.1.3. Bovespa intraday returns

These data, which cover 305 trading days from September 1, 2009, to November 6, 2010, were made available by Capse Investimentos. This is exactly the data set that was used by Horta and Ziegelmann (2018). The tick-by-tick series is sampled at 5-minute intervals, such that we have a sample \((p_{i,t}, i = 1, \ldots, m_t + 1)\) for each day \(t = 1, \ldots, n\), where \(m_t + 1\) is the number of log-price observations within day \(t\). We define the \(i\)th 5-minute return on day \(t\) as

\[
r_{i,t} := p_{i+1,t} - p_{i,t}, \quad i = 1, \ldots, m_t.
\]  

For each day \(t\), the 5-minute returns in Eq. (12) are distributed according to some density \(f_t\), which we estimate as is explained in Section 3.2. Examples of such densities are given in Fig. 3.

3.1.4. XLK intraday returns

These intraday returns are constructed in the same way as the Bovespa returns but the underlying asset is XLK, the Technology Select Sector SPDR Fund. The time period is exactly the same as for the Bovespa data.

### Fig. 2.
Kernel density estimates with a Gaussian kernel and bandwidth selected by Silverman’s rule of thumb for the monthly Dow-Jones index from April 2004 to December 2017. The solid black line shows a density function of one representative month that has mostly negative returns; the red dashed line shows a density function of one representative month that has mostly positive returns; the blue dash-dotted line shows a density function of one representative month that has returns close to zero.

### Fig. 3.
Three density estimates, obtained in the same way as the estimates in Fig. 2, for the Bovespa intraday return. These densities are leptokurtic and are generally centered around zero.

#### 3.2. Density estimation

The true densities are not observable. We work with densities which are outputs of kernel estimators:

\[
f_t(u) = \frac{1}{n_t h_t} \sum_{i=1}^{n_t} K \left( \frac{u - r_{i,t}}{h_t} \right), \quad t = 1, \ldots, n,
\]

where \(K(\cdot)\) is a kernel and \(h_t\) is a bandwidth. We consider two commonly-used kernels, Gaussian and Epanechnikov, given by

\[
K(u) = \left( \sqrt{2\pi} \right)^{-1} \exp\left(-u^2/2\right), \quad \text{(Gaussian kernel)}
\]

\[
K(u) = \frac{3}{4} (1 - u^2)^2, \quad |u| \leq 1. \quad \text{(Epanechnikov kernel)}
\]

where \(\hat{\sigma}_t\) denotes the sample standard deviation of the returns \(r_{i,t}, i = 1, \ldots, n_t\). We also consider the direct plug-in (DPI) method of Sheather and Jones (1991) for bandwidth selection.

We compare the predicted future density \(f_{n+h}\), with its estimate \(f_{n+h}\), obtained using the above methods. Section 3.3 explains the measures of the discrepancy that we use.

#### 3.3. Measures of forecast accuracy

We measure the difference between the forecast density and the estimated future density by considering the discrete version of the Kullback–Leibler divergence (KLD; see Kullback & Leibler, 1951), the square root of the Jensen–Shannon divergence (JSD; see Shannon, 1948), and the mean \(L_p\)-norms with \(p = 1, 2, \infty\). The KLD is designed to measure the loss of information when we
choose an approximation. For two probability density functions, denoted by \( g_t(u) \) and \( \hat{g}_t(u) \), the discrete version of the KLD is given by

\[
\text{KLD} = D_{KL}(g_t \parallel \hat{g}_t) + D_{KL}(\hat{g}_t \parallel g_t) = \sum_{i=1}^{N} g_t(u_i) \cdot [\ln g_t(u_i) - \ln \hat{g}_t(u_i)] + \sum_{i=1}^{N} \hat{g}_t(u_i) \cdot [\ln \hat{g}_t(u_i) - \ln g_t(u_i)],
\]

which is symmetric and nonnegative.

An alternative is given by the JSD, defined as

\[
\text{JSD} = \frac{1}{2} D_{KL}(g_t \parallel \delta_t) + \frac{1}{2} D_{KL}(\hat{g}_t \parallel \delta_t),
\]

where \( \delta_t \) measures a common quantity between \( g_t \) and \( \hat{g}_t \). We consider the simple mean and the geometric mean, given by \( \delta_t = \frac{1}{2} (g_t + \hat{g}_t) \) or \( \delta_t = \sqrt{g_t \hat{g}_t} \). The JSD is locally proportional to the Fisher information metric, and is similar to the Hellinger metric, in the sense that it induces the same affine connection on a statistical manifold, and is equal to half the so-called Jeffreys divergence. We make the JSD a metric between any two probability densities by taking its square root (see e.g. Fuglede & Topsøe, 2004).

Let \( \hat{e}_t(x) \) be the difference between \( g_t(x) \) and \( \hat{g}_t(x) \); then, the \( L_1 \)-norm, \( L_2 \)-norm and \( L_\infty \)-norm are

\[
|\hat{e}_t|_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} \left| \sum_{t=1}^{N} \hat{e}_t(x_i) \right|,
\]

\[
|\hat{e}_t|_2 = \frac{1}{n_1} \left( \sum_{i=1}^{n_1} \left( \sum_{t=1}^{N} \hat{e}_t(x_i) \right)^2 \right)^{1/2},
\]

\[
|\hat{e}_t|_\infty = \frac{1}{n_1} \sum_{i=1}^{n_1} \max_{t} |\hat{e}_t(x_i)|,
\]

where \( n_1 \) denotes the forecasting period.

3.4. Tables of forecast accuracy measures

We use the expanding window approach, which is fairly standard for comparisons of this type (see e.g. Zivot & Wang, 2006, Chapter 9). Parameter estimates and their forecasts are computed over an expanding window, with the start point being at the beginning of the sample and the end point starting at the end of the first sample, which is of a reasonable length, and moving forward one step at a time until it reaches the end of the sample. This gives one fewer forecasts than end points.

For the Dow Jones and S&P 500 data, which cover the same period of time, we proceed as follows. We use the first 110 density estimates, from April 2004 to May 2013, to produce a one-step-ahead forecast of the June 2013 density, which we treat as an unknown future density. Next, using an expanding-window approach, we re-estimate the parameters in the density forecasting methods using the first 111 estimated densities, from April 2004 to June 2013, and compute the one-step-ahead forecast of the density with monthly index 112. We iterate this process by increasing the sample size by one month at a time until we reach the end of the data period in December, 2017. This process produces 55 one-step-ahead forecasts, which we compare to the actual density estimates. Note that we do not observe true densities in practice, but can construct density estimates from the available data.

Table 1 compares the density estimation accuracies of the four methods for the Dow-Jones cross-sectional returns. Of the two kernel functions, it is generally advantageous to use the Epanechnikov kernel function because of the superiority of its estimation accuracy over that of the Gaussian kernel function. Of the two bandwidth selection methods, it is better to use the rule-of-thumb because of its superior estimation accuracy and high computational speed relative to the plug-in method. The CoDa method performs the best as measured by the Kullback–Leibler divergence and the Jensen–Shannon divergence with the geometric mean, while the log quantile density transformation method performs the best as measured by the Jensen–Shannon divergence with the simple mean, \( L_1 \)-norm, \( L_2 \)-norm and \( L_\infty \)-norm. The difference in accuracy between the CoDa methods with and without standardization is small, though the CoDa method without standardization has a slight edge.

Table 2 compares the density estimation accuracies of the four methods for the S&P 500 cross-sectional returns. Of the two kernel functions, it is better to use the Epanechnikov kernel function because of its superior estimation accuracy relative to the Gaussian kernel function. Of the two bandwidth selection methods, it is better to use the rule-of-thumb because of its superior estimation accuracy and fast computational speed relative to the plug-in method. As measured by the Jensen–Shannon divergence with a simple mean, the log quantile density transformation method generally performs the best, while the CoDa method performs the best as measured by the remaining criteria. The difference in accuracy between the CoDa methods with and without standardization is small.

Regarding the Bovespa and XLK data, we start with the first 203 densities from September 1, 2009, to June 30, 2010. We then increase the sample size by one day at a time until we reach the end of the data period on November 26, 2010. This process produces 102 one-step-ahead forecasts, which we compare to the actual density estimates.

Table 3 compares the density estimation accuracies of the four methods for the Bovespa intraday returns. Of the two kernel functions, it is generally better to use the Epanechnikov kernel function because of its superior estimation accuracy relative to the Gaussian kernel function. Of the two bandwidth selection methods, it is better to use the rule-of-thumb because of its superior estimation accuracy and fast computational speed relative to the plug-in method. The log quantile density transformation method generally performs the best as measured by the Jensen–Shannon divergence with a simple mean, while the CoDa method generally performs the best as measured by the Kullback–Leibler divergence. The difference in accuracy between the CoDa methods with and without standardization is small.

Table 4 compares the density estimation accuracies of the four methods for the XLK intraday returns. Of the
### Table 1
Forecast accuracies of the four methods, Dow-Jones cross-sectional returns.

<table>
<thead>
<tr>
<th>Parameter selection</th>
<th>Density forecast method</th>
<th>KLD</th>
<th>JSD norm</th>
<th>Simple</th>
<th>Geometric</th>
<th>norm L₁</th>
<th>L₂</th>
<th>L∞</th>
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<td></td>
</tr>
<tr>
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<td>46.96</td>
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</tr>
<tr>
<td></td>
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<tr>
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<td><strong>3.62</strong></td>
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<td>16.61</td>
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<tr>
<td></td>
<td>CoDa (no standardization)</td>
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<td>16.81</td>
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Note: The smallest error for each choice of kernel function, bandwidth selection method and evaluation criterion is highlighted in bold.

### Table 2
Forecast accuracies of the four methods, S&P 500 cross-sectional returns.

<table>
<thead>
<tr>
<th>Parameter selection</th>
<th>Density forecast method</th>
<th>KLD</th>
<th>JSD norm</th>
<th>Simple</th>
<th>Geometric</th>
<th>norm L₁</th>
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<td><strong>Gaussian kernel</strong></td>
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<tr>
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two kernel functions, it is better to use the Epanechnikov kernel function because of its superior estimation accuracy relative to the Gaussian kernel function. Of the two bandwidth selection methods, it is better to use the rule of thumb because of its superior estimation accuracy and high computational speed relative to the plug-in method. When the kernel function is the Epanechnikov kernel, the log quantile density transformation method generally performs the best. However, when the kernel function is the Gaussian kernel, the skewed-t distribution produces the smallest Kullback–Leibler divergence and Jensen–Shannon divergence with the geometric mean, while the log quantile density transformation method produces the smallest errors as measured by the Jensen–Shannon divergence with the simple mean and almost all of the distance metrics.
Table 3
Forecast accuracies of the four methods, Bovespa intraday returns.

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Table 4
Forecast accuracies of the four methods, XLK intraday returns.

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3.5. Model confidence set

The model confidence set procedure proposed by Hansen, Lunde, and Nason (2011) consists of a sequence of tests that permit the construction of a set of “superior” models for which the null hypothesis of equal predictive ability (EPA) is not rejected at a specified confidence level. The EPA test statistic can be evaluated for any arbitrary loss function. The model confidence set (MCS) procedure is a sequential testing procedure that eliminates the worst model at each step until the hypothesis of equal predictive ability is accepted for all of the models that belong to a set of superior models. The selection of the worst model is determined by an elimination rule that is consistent with the test statistic. This paper uses the \( T_{max} \) test statistic, which is a default statistic in the \( R \) package MCS and performed well in the study by Shang and Haberman (2018). With the bandwidth being selected by the rule-of-thumb...
and for both kernel functions, we use the MCS procedure to select a set of superior models for the four data sets. For each of the five error criteria, Table 5 indicates the superior set of models with daggers, †.

Table 5 can be summarized as follows. With four data sets, two different kernel functions and six forecast error criteria, there are 48 cases. The numbers of cases in which each specific method is superior are as follows: (1) LQDT: 24; (2) CoDa (no standardization): 14; (3) CoDa (standardization): 13; (4) dynamic FPCR: 5; (5) skewed-t: 2. Differentiating between the two types of densities, we see that the LQDT and CoDa methods are both recommended for the monthly cross-sectional returns. For the daily intraday returns, the LQDT is recommended.

3.6. Conclusions

We have compared the performances of five methods using four raw data sets, from which eight data sets consisting of densities have been constructed. These eight data sets cannot lead to a complete picture, but they already show a fairly clear pattern. With the exception of a focus on the densities of returns in finance, the data sets have not been selected using any specific criteria, and we have not experimented with any other datasets, so the results presented in Section 3.4 are not a selection of “favorable” results. We used six forecast quality criteria, meaning that each of the five methods is compared in

We have compared the performances of five methods using four raw data sets, from which eight data sets consisting of densities have been constructed. These eight data sets cannot lead to a complete picture, but they already show a fairly clear pattern. With the exception of a focus on the densities of returns in finance, the data sets have not been selected using any specific criteria, and we have not experimented with any other datasets, so the results presented in Section 3.4 are not a selection of “favorable” results. We used six forecast quality criteria, meaning that each of the five methods is compared in 8 × 6 = 48 scenarios.

The predictions are almost always better when we use ROT rather than DPI, so our summary of the results in Section 3.4 focuses on the criteria computed for the bandwidth selected by ROT. The LQDT method is the best in 25 of these scenarios, while one of the two CoDa methods is the best in 17 scenarios. The CoDa (no standardization) works a little better. The performance of the Horta-Ziegelmann method is practically the same as the CoDa methods; it is either the best or the second best in 50% of cases. For some data, the skewed-t method performs well according to the KLD criterion. In many cases, the differences between the measures of forecast accuracy are small. However, a pattern that has emerged suggests that the LQDT method, which has been developed both theoretically and numerically in this paper, should be used as the first choice for the purpose of the prediction of cross-sectional and intraday returns. This is confirmed further by the MCS analysis in Section 3.5, where we noted that the LQDT and CoDa methods are both recommended for the monthly cross-sectional returns.

The rankings of the methods may be different for other time series of densities, e.g. those arising in population studies. It is hoped that the algorithmic descriptions and background presented in this paper will allow researchers in fields other than finance to assess the usefulness of these forecast methods.

Acknowledgments

The authors acknowledge the insightful comments and suggestions made by an Associate Editor and two reviewers; they led to a much-improved manuscript. Han Lin Shang would like to thank for valuable comments made by an Associate Editor and two reviewers; they led to a much-improved manuscript. Han Lin Shang would like to thank for valuable comments made by an Associate Editor and two reviewers; they led to a much-improved manuscript.

Appendix A. Theoretical justification of the LQDT method of Section 2.2

This section defines the class of densities for which the transformations defined in Section 2.2 are well defined, and shows that the inverse transformation produces a density.

Definition A.1. Let $D$ be the space of densities with support $(r^-, r^+]$, $-\infty < r^- < 0 < r^+ < \infty$, such that $f$ is continuous on $[r^-, r^+]$ for all $f \in D$, and either $f(r) > 0$ on $[r^-, r^+]$ or there exist constants $c_j, \gamma_j > 0$ and $r_j \in (r^-, r^+]$ such that

\[ c_0(r - r^-)^{\gamma_0} \leq f(r) \leq c_1(r - r^-)^{\gamma_1}, \quad r^- < r < r_0, \]

\[ c_2(r^-)^{\gamma_2} \leq f(r) \leq c_3(r^+)^{\gamma_3}, \quad r_1 < r < r^+. \]

Proposition A.1. Suppose that $f \in D$. Then, $Y$ defined by Eq. (8) is in $L^2[0, 1]$. Furthermore, let $c \in (0, 1)$ and let $g$ be a continuous function on $[0, 1)$. Then $Q_{g, c}(s) := \int_0^s \exp(g u) du$ is continuous and increasing on $(0, 1)$, it has a well-defined inverse $F_{g, c} = Q_{g, c}^{-1}$, and the inverse log quantile density transformation given by

\[ f_{g, c}(r) := \psi^{-1}(g, c)(r) \]

\[ = \begin{cases} \exp\{-g \circ F_{g, c}(r)\}, & -\infty \leq Q_{g, c}(0) < r < Q_{g, c}(1) \leq \infty, \\ 0, & \text{otherwise}, \end{cases} \]

is well-defined and satisfies $F'_{g, c}(r) = f_{g, c}(r) \geq 0, F_{g, c}(0) = c$, and $\int_{-\infty}^{Q_{g, c}(1)} dr = 1$.

Proof. Clearly, $Y \in L^2[0, 1]$ if $f$ is strictly positive in its support $[r^-, r^+]$. Otherwise, for small $\epsilon > 0$,

\[ \int_0^s Y^2(s)ds = \int_{r^-}^{r^+} \log^2(f(r))f(r)dr \quad (r = Q(s)) \]

\[ = c_1 \int_{r^-}^{r^+} \left[ \log(c_0) + \gamma_0 \log(r - r^-) \right]^2 (r - r^-)^{\gamma_1} dr \quad (w = -\log(r - r^-)) \]

\[ = c_1 \int_{-\infty}^{\log(c_0)} \left[ \log(c_0) - \gamma_0 w \right]^2 e^{-(\gamma_1 + 1)w} dw < \infty. \]

A similar argument shows that $\int_{r^-}^{r^+} Y^2(s)ds < \infty$.

Next, take $g, c, Q_{g, c}, F_{g, c}$, and $f_{g, c}$ as in the statement of the proposition, so that $f_{g, c} \geq 0$ is clear. As $Q_{g, c}(c) = 0$, we also have $F_{g, c}(0) = c$. The limits

\[ Q_{g, c}(0) := \lim_{s \to 0} Q_{g, c}(s), \quad Q_{g, c}(1) := \lim_{s \to 1} Q_{g, c}(s) \]

are well-defined, since $Q_{g, c}$ is increasing, where we may have $Q_{g, c}(0) = -\infty, Q_{g, c}(1) = \infty$, or both. Taking the
Change of variables $s = F_{g,c}(u)$, we have $u = Q_{g,c}(s)$ and $du = e^{g(s)}ds$, so that for $Q_{g,c}(0) \leq r \leq Q_{g,c}(1)$,

$$
\int_{-\infty}^{\infty} f_{g,c}(u)du = \int_{Q_{g,c}(0)}^{Q_{g,c}(1)} \exp \{ -g \circ F_{g,c}(u) \} du = \int_{0}^{F_{g,c}(r)} e^{-g(s)}e^{g(s)}ds = F_{g,c}(r),
$$
i.e., $F_{g,c}^{-1}(r) = F_{g,c}(r)$. Plugging in $r = Q_{g,c}(1)$ proves that $F_{g,c}$ is indeed a density with support $(Q_{g,c}(0), Q_{g,c}(1))$.

Lastly, we verify that this is truly the inverse. For a density $f \in \mathcal{D}$ with cdf $F$, quantile function $Q$, and $q = Q'$, set $(g, c) = (\psi(f), c)$, so $Q(c) = 0$ and $g(s) = -\log(f \circ Q(s))$ is continuous on $(0, 1)$. Then, as $q(c) = 1/(f \circ Q(c))$ by Eq. (6),

$$
Q_{g,c}(s) = \int_{c}^{s} e^{g(u)}du = \int_{c}^{s} \exp \{ -\log(f \circ Q(u)) \} du = \int_{c}^{s} q(u)du = Q(u).
$$

As a result, $F_{g,c} = F$ and, for $r \in (r^-, r^+) = (Q(0), Q(1))$,

$$
f_{g,c}(r) = \exp \{ -g \circ F_{g,c}(r) \} = \exp \{ \log(f \circ Q \circ F(r)) \} = f(r),
$$
i.e., $\psi^{-1} \circ \psi(f) = f$.

In the other direction, if $g$ is continuous on $(0, 1)$ and $c \in (0, 1)$, set $f_{g,c} = \psi^{-1}(g, c)$ to be the inverse-mapped density with cdf $F_{g,c}$ and quantile function $Q_{g,c} = F_{g,c}^{-1}$. The log quantile density transformation of $F_{g,c}$ is $Y, F_{g,c}(0) =...
\[ \psi(f_{g,c}), \text{ where} \]
\[ Y(s) = -\log(f_{g,c} \circ Q_{g,c}(s)) \]
\[ = -\log \left( \exp \left\{ -g \circ f_{g,c} \circ Q_{g,c}(s) \right\} \right) = g(s) \]
and \( f_{g,c}(0) = c, \) i.e., \( \psi \circ \psi^{-1}(g, c) = (g, c). \)

**Appendix B. Log hazard transformation**

This section shows that the log hazard transformation in Eq. (B.1) maps densities into the space \( L^2 \) under weak assumptions, but the inverse transformation does not necessarily transform the predicted \( L^2 \) curves into densities. Define

\[ Y_t(r) = \psi(f_t X(r)) = \begin{cases} \log \left( \frac{f_t(r)}{1 - f_t(r)} \right), & \text{if } f_t(r) > 0, \\ 0, & \text{if } f_t(r) = 0. \end{cases} \]  
(B.1)

The following conditions are sufficient to ensure \( Y_t \in L^2(-\infty, \infty) \), and are not too strong. Recall that we assume that \( f_t \) is continuous and that Eq. (7) holds. Suppose that

\[ f_t(r) \geq c_0(r - r_t^-)^{\gamma_1}, \quad r_t^- < r < r_0, \]
\[ c_1(r_t^+ - r)^{\gamma_1} \leq f_t(r) \leq c_2(r_t^+ - r)^{\gamma_2}, \quad r < r < r_t^+, \]

for some \( c_1, \gamma_1 > 0 \) and \( r_t^+ \). Then, for small \( r - r_t^- \),

\[ |Y_t(r)| = \left| \log \left( \frac{1 - F_t(r)}{f_t(r)} \right) \right| \leq -\log(f_t(r)) \]
\[ \leq \log(c_0^{-1}) - \gamma_0 \log(r - r_t^-). \]

Since \( \int_{r_t^-}^{r_t^+} \log^2(s) ds < \infty \), for small \( \epsilon \),

\[ \int_{r_t^-}^{r_t^+} Y_t^2(s) ds \leq \int_{r_t^-}^{r_t^+} \left[ \log(c_0) + \gamma_0 \log(s - r_t^-) \right]^2 ds < \infty. \]

Similarly, for \( r_t^+ - r \) small, \( 1 - F_t(r) \leq c_1(\gamma_1 + 1)^{-1}(r_t^+ - r)^{-\gamma_1} = c_1(r_t^+ - r)^{-\gamma_1} \), so that

\[ |Y_t(r)| = \left| \log \left( \frac{f_t(r)}{1 - F_t(r)} \right) \right| \leq \log(c_2/c_1) + (\gamma_2 - \gamma_1) \log(r_t^+ - r), \]

and \( \int_{r_t^-}^{r_t^+} Y_t^2(s) ds < \infty \). Thus, \( Y_t \in L^2(-\infty, \infty) \).

Suppose that \( g \) is a continuous function with sup(g) \( = [a, b] \). In our prediction context, \( g = Y_{n+h} \).

\[ A_g(x) = \int_a^x e^{g(y)} dy, \quad x \in (a, b), \]

and

\[ \psi^{-1}(g(x)) = \exp \{ g(x) - A_g(x) \}, \]

denote \( f(x) = \psi^{-1}(g(x)) \). Then, \( f \geq 0 \) and

\[ \int_a^b f(x) dx = \int_0^{A_g(b)} e^{-u} du = 1 - e^{-A_g(b)}. \]

Thus, for \( f \) to be a density, we would need \( A_g(b) = \infty \), which could be understood as a density as \( x \uparrow b \); i.e., we would need

\[ \lim_{x \uparrow b} \int_a^x e^{g(y)} dy = \infty. \]  
(B.2)

However, Eq. (B.2) is not necessarily true in general; for example, if \( \lim_{x \uparrow b} g(x) = g(b) < \infty \) Eq. (B.2) holds if \( g(y) = \log \{ h(y) \} \) for some density \( h \).

An additional issue is that \( Y_t \) necessarily diverges at \( r_t^- \) and \( r_t^+ \), which are random and differ for each \( t \), making \( Y_t \) an ill-behaved stochastic process.

**References**


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