Change point and trend analyses of annual expectile curves of tropical storms

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Joint work with

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Colorado State University
Outline of the talk:

1. Expectile curves of tropical storm strength
2. A functional regression model
3. Change point and trend tests
4. Application to hurricane and typhoon data
<table>
<thead>
<tr>
<th>Time per Year</th>
<th>Typhoons in 2005</th>
<th>Hurricanes in 2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td></td>
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<td>Mar</td>
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<td>May</td>
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<td>Sep</td>
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<td>Nov</td>
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**Trend in expectile curves**

Expectile curves for $\tau = 0.1, 0.5$ and $0.9$.
Expectiles, background

$Y$ is a square integrable random variable

\[
E_-(e) = E \left[ (X - e)^2 \mathbb{I}\{X \leq e\} \right]; \\
E_+(e) = E \left[ (X - e)^2 \mathbb{I}\{X > e\} \right].
\]

\[
E_\tau(e) = (1 - \tau)E_-(e) + \tau E_+(e),
\]

The $\tau$th expectile $e_\tau$ minimizes $E_\tau(e)$.

($\tau$ close to 1, $E_+(e)$ must be small, $e_\tau$ must be large.)

Estimation through empirical expectations.

Idea can be extended to the scatter plot of points $(t_i, x_i)$, $t_i \in I$.

Our application: $I = \text{year}$, $x_i$ wind speed at time $t_i$.

Using spline smoothing, one obtains expectile curves.
change point test

We observe curves $X_1, X_2, \ldots, X_N$, $\mu_i(t) = \operatorname{E}X_i(t)$.

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_N.$$  

Test based on projections of

$$P_k(t, \tau) = \frac{k(N - k)}{N} \left\{ \hat{\mu}_k(t, \tau) - \tilde{\mu}_k(t, \tau) \right\},$$

Scores:

$$\hat{\xi}_{j,n} = \int \left\{ X_n(t) - \bar{X}_N(t) \right\} \hat{v}_j(t) dt.$$  

Test statistics:

$$\hat{S}_d = \frac{1}{N^2} \sum_{j=1}^d \frac{1}{\hat{\lambda}_j} \sum_{k=1}^N \left( \sum_{1 \leq i \leq k} \hat{\xi}_{j,i} - \frac{k}{N} \sum_{1 \leq i \leq k} \hat{\xi}_{j,i} \right).$$

Limit distribution of Kiefer’s type 
(sum of integrals of squared Brownian bridges).

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Trend in expectile curves
$P_n(t; \tau)$ functions, typhoons

Trend in expectile curves
Trend model

\[ X_n(t) = \alpha(t) + \beta(t)n + \varepsilon_n(t). \]

\[ H_0 : \beta = 0, \quad \text{vs.} \quad H_A : \beta \neq 0. \]

LSE:

\[ \hat{\beta}(t) = \frac{6}{N(N+1)(N-1)} \sum_{k=1}^{N} (2k - N - 1)X_k(t) \]

If the function \( \hat{\beta} \) is large, we reject \( H_0 \).

\( \lambda_j \) eigenfunctions of the error curves \( \varepsilon_i \);
\( \hat{\lambda}_j \) sample eigenvalues of the residual curves \( \hat{\varepsilon}_i \);
\( \hat{\nu}_j \) corresponding eigenfunctions (sample FPC’s)
Hurricanes coefficient functions

$\tau = 0.6$

\[\begin{array}{cccccccccccccc}
\text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \\
-0.4 & 0.2
\end{array}\]

$\tau = 0.7$

\[\begin{array}{cccccccccccccc}
\text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \\
-0.4 & 0.2
\end{array}\]

$\tau = 0.8$

\[\begin{array}{cccccccccccccc}
\text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} & \text{Jun} & \text{Jul} & \text{Aug} & \text{Sep} & \text{Oct} & \text{Nov} & \text{Dec} \\
-0.4 & 0.2
\end{array}\]
Typhoons coefficient functions

\[ \tau = 0.6 \]

\[ \tau = 0.7 \]

\[ \tau = 0.8 \]

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Trend in expectile curves
Norms of the slope functions $\hat{\beta}$

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**Typhoons**

**Hurricanes**

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Trend tests

Monte Carlo test:
Under $H_0$,

$$\hat{\Lambda}_N = \frac{N^3}{12} \int_0^1 \left( \hat{\beta}(t) \right)^2 dt \xrightarrow{\mathcal{L}} \Lambda_{\infty} \overset{\text{def}}{=} \sum_{j=1}^{\infty} \lambda_j Z_j^2,$$

($Z_j$ independent standard normal)

Chi–square test:
Under $H_0$,

$$\hat{T}_N = \frac{N^3}{12} \sum_{j=1}^{q} \hat{\lambda}_j^{-1} \left( \hat{\beta}, \hat{v}_j \right)^2 \xrightarrow{\mathcal{L}} \chi_q^2.$$

Both tests are shown to be consistent.
Application to typhoon and hurricane data

Change point test rejects $H_0$ for all values of $\tau \in .1, .2, \ldots, .9$

### Monte Carlo test P–values

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>typhoons</td>
<td>0.365</td>
<td>0.537</td>
<td>0.545</td>
<td>0.495</td>
<td>0.438</td>
<td>0.381</td>
<td>0.329</td>
<td>0.309</td>
<td></td>
</tr>
<tr>
<td>hurricanes</td>
<td>0.439</td>
<td>0.239</td>
<td>0.133</td>
<td>0.081</td>
<td>0.062</td>
<td>0.047</td>
<td>0.038</td>
<td>0.029</td>
<td></td>
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</table>

### Chi–square test P–values

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<th>$\tau$</th>
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<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>typhoons</td>
<td>0.534</td>
<td>0.705</td>
<td>0.722</td>
<td>0.688</td>
<td>0.587</td>
<td>0.466</td>
<td>0.382</td>
<td>0.337</td>
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<tr>
<td>hurricanes</td>
<td>0.069</td>
<td>0.024</td>
<td>0.015</td>
<td>0.006</td>
<td>0.003</td>
<td>0.003</td>
<td>0.004</td>
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Trend in expectile curves
Main conclusions

- Simulations show that the Monte Carlo test is more accurate if DGP’s resemble actual data.
- The annual pattern of wind speeds of both hurricanes and typhoons has been changing at all wind speed levels over the last 60 years.
- There is a significant trend in the shape of this pattern for upper wind speed levels of hurricanes.