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Estimation of the Long-Run Variance of Nonlinear Time Series With an Application to Change Point Analysis

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ABSTRACT

For a broad class of nonlinear time series known as Bernoulli shifts, we establish the asymptotic normality of the smoothed periodogram estimator of the long-run variance. This estimator uses only a narrow band of Fourier frequencies around the origin and so has been extensively used in local Whittle estimation. Existing asymptotic normality results apply only to linear time series, so our work substantially extends the scope of the applicability of the smoothed periodogram estimator. As an illustration, we apply it to a test of changes in mean against long-range dependence. A simulation study is also conducted to illustrate the performance of the test for nonlinear time series.

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1 | Introduction

Over the past two decades, classical temporal dependence conditions based on cumulants or mixing coefficients have been complemented by conditions formulated in terms of nonlinear moving averages of the form

$$X_t = g(\varepsilon_t, \varepsilon_{t-1}, \dots), \quad t \in \mathbb{Z} \tag{1}$$

where g can be a fairly general function. Among them, the *Physical or Functional Dependence* and the *Approximability* have gained particular popularity and have been used in dozens of publications. Physical dependence was introduced by Wu (2005) and studied or assumed in Wu and Mielniczuk (2010), Liu et al. (2013), Zhou (2013), Zhang and Wu (2017), Dette et al. (2020) and van Delft (2020), among many others. Approximability was also used in a large number of papers, for example, in Aue et al. (2009), Hörmann and Kokoszka (2010), Berkes et al. (2013), Horváth et al. (2013), Horváth et al. (2014), Kokoszka et al. (2015), Zhang (2016), Bardsley et al. (2017), and Horváth et al. (2024).

Estimation of the spectral density function, which includes estimation of the long-run variance (LRV) is a key problem in time series analysis that has been studied for several decades. The LRV of a scalar time series is introduced in most time series textbooks, for example, in Hamilton (1994). Multivariate, high-dimensional, and functional analogs of the long-run variance have been extensively investigated. The contributions of Wu and Xiao (2012), Berkes et al. (2016), Chan (2022) and Baek et al. (2023), among many others, provide many references. McElroy and Politis (2024) explain why estimation of the spectral density at frequencies near zero requires care and propose effective solutions.

Specialized to the estimation of the LRV, the lag window estimator,

$$\hat{f}(0) = \frac{1}{2\pi} \sum_{|r| \le n-1} w_n(r) \hat{\gamma}(r) \quad (2)$$

where the $w_n(r)$ are weights such that $w_n(r) \to 0$, as $|r| \to \infty$, and the $\hat{\gamma}(r)$ are the usual sample autocovariances, has been particularly extensively studied. Anderson (1971) developed its theory for linear processes, while Rosenblatt (1984) considered strong mixing processes satisfying cumulant summability conditions. These results were extended to nonlinear time series. Chanda (2005) assumed a specific form of the general Bernoulli shifts (1), namely (for a fixed w), $X_t = \epsilon_t + \sum_{r=1}^{\infty} g_r(\epsilon_{t-1}, \dots, \epsilon_{t-r-w})$, with summability conditions on the fourth moments of the $g_r(\epsilon_{t-1}, \dots, \epsilon_{t-r-w})$. The above representation is motivated by a truncated Volterra expansion. Results of Shao and Wu (2007b), Wu and Shao (2007), and Liu and Wu (2010), which assume only the general representation (1), can be specialized to yield asymptotic normality of the estimator (2) under the additional assumption that the lag windows in Equation (2) are of the scale parameter form (i.e., $w_n(r) = w(r/B_n)$, where w is a kernel and B_n is a bandwidth) and under various conditions related to the physical dependence of Wu (2005).

In this article, we consider the smoothed periodogram estimator

$$Q_n = \frac{1}{m} \sum_{j=1}^{m} I_n(\omega_j) \quad (3)$$

where $I_n(\omega_j)$ is the periodogram at the Fourier frequency ω_j and m is the count of frequencies used in the estimation ($m/n \to 0$). While the lag window estimator (2) can be expressed as a weighted sum of the periodogram over *all* Fourier frequencies, estimator (3) uses only a narrow band of local Fourier frequencies. For this reason, it is particularly relevant in local Whittle estimation, see, for example, Robinson (1995), Velasco (1999), Phillips and Shimotsu (2004), Robinson (2008), Baek and Pipiras (2012), Li et al. (2021), and Baek et al. (2024).

To the best of our knowledge, the asymptotic distribution of estimator (3) is only available under the assumption of linearity, see Giraitis et al. (2012) for a comprehensive account. The estimator (3) can be alternatively expressed as the lag window estimator (2) with $w_n(0) = 1$ and

$$w_n(r) = \frac{1}{2m} \left[\frac{\sin((2m+1)\pi r/n)}{\sin(\pi r/n)} - 1 \right] \quad (4)$$

for $0 < |r| < n - 1$. The $w_n(r)$ in Equation (4) are not of the scale parameter form, and the asymptotic distribution of Q_n does not follow from the currently available results for nonlinear time series.

A related but different estimator of the LRV is Daniell's estimator given by (see, e.g., Chapter 9 of Anderson 1971)

$$\frac{n/m}{\pi} \int_0^{\pi/(n/m)} I_n(\omega)\, d\omega = \frac{1}{2\pi} \sum_{r=-(T-1)}^{T-1} \frac{\sin(\pi r/(n/m))}{\pi r/(n/m)} \hat{\gamma}(r) \quad (5)$$

with the convention that $\sin(\pi r/(n/m))/(\pi r/(n/m)) = 1$ when $r = 0$. Daniell's lag window $w(r) = \sin(\pi r)/(\pi r)$ is of the scale parameter form, so the asymptotic normality of Daniell's estimator for nonlinear time series follows directly from the currently available results (see Liu and Wu 2010, Theorem 2), unlike the asymptotic normality of the estimator Q_n.

In this article, we establish the asymptotic normality of Q_n for general nonlinear moving averages and present an application to change point analysis. We note that Brillinger (1969) considers an estimator similar to (3) (see his equation (6.6)), but for a fixed m, and (3) is not even consistent then. Our proofs are based on a general central limit theorem for quadratic forms established by Liu and Wu (2010), which we state as Theorem 8.

The relationship between the lag window estimator at any frequency ω and the periodogram ordinates is explained in Section 6.2.3 of Priestley (1981) and Section 3 of Shao and Wu (2007b), among others. To emphasize the difference, we note that the lag window estimator can be expressed as $\hat{f}(\omega) = \int_{-\pi}^{\pi} K_n(\omega - \lambda) I_n(\lambda) d\lambda$, so it involves all frequencies. On the other hand, the quadratic form of the lag window estimator involves only the terms $X_t X_\tau$ with $|t - \tau| \le \ell_n$, whereas the smoothed periodogram involves these terms with all indexes t, τ. These differences explain why different techniques of proof must be used for these two classes of estimators.

The paper is organized as follows. In Section 2, we introduce the objects we study. Section 3 contains the main theorems, while Sections 4 and 5 present, respectively, a theoretical application to change point analysis and a related, small simulation study. All proofs are collected in Section 6, which also contains a central limit theorem for weighted sums of the periodogram ordinates that we use in our proofs, and that is also of independent interest. Graphs and Tables related to the simulation study are presented in the online Supporting Information.

2 | Preliminaries

2.1 | The Spectral Density and the Periodogram

We assume in the following that $\{X_t\}_{t \in \mathbb{Z}}$ is a strictly stationary sequence of real random variables satisfying

$$\mathrm{E}X_1 = 0 \quad \text{and} \quad \mathrm{E}X_1^2 < \infty$$

Let $\{\gamma(h)\}_{h \in \mathbb{Z}}$ be the sequence of autocovariances of $\{X_t\}_{t \in \mathbb{Z}}$, where $\gamma(h) = \mathrm{Cov}(X_h, X_0)$ for $h \in \mathbb{Z}$. The spectral density function of the series $\{X_t\}_{t \in \mathbb{Z}}$ is defined by

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\omega}, \quad \omega \in [-\pi, \pi] \quad (6)$$

The discrete Fourier transform (DFT) of $X_1, \dots, X_n$ and the periodogram are defined by

$$\mathcal{X}_n(\omega_j) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t e^{-it\omega_j} \quad \text{and} \quad I_n(\omega_j) = |\mathcal{X}_n(\omega_j)|^2 \quad (7)$$

with the Fourier frequencies $\omega_j = 2\pi j/n$, where $j \in \{-\lfloor (n-1)/2 \rfloor, \dots, \lfloor n/2 \rfloor\}$ and $\lfloor \cdot \rfloor$ is the floor function.

2.2 | Weak Dependence

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a sequence of real random variables of the form (1), where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of independent and identically distributed (iid) random elements in a measurable space S and $g : S^\infty \to \mathbb{R}$ is a measurable function. Series of the form (1) are often called Bernoulli shifts and are automatically strictly stationary.

Suppose that $\{\varepsilon'_t\}_{t\in\mathbb{Z}}$ is an independent copy of $\{\varepsilon_t\}_{t\in\mathbb{Z}}$. Define $X_{t,\{0\}}$ by replacing ε_0 in Equation (1) by ε'_0, that is,

$$X_{t,\{0\}} = g(\varepsilon_t, \ldots, \varepsilon_1, \varepsilon'_0, \varepsilon_{-1}, \ldots), \quad t \in \mathbb{Z} \tag{8}$$

We write $X \in \mathcal{L}^p$ with $p > 0$ if $\|X\|_p = (\mathrm{E}|X|^p)^{1/p} < \infty$. The physical dependence measure is defined by

$$\delta_{t,p} = \|X_t - X_{t,\{0\}}\|_p, \quad t \in \mathbb{Z} \tag{9}$$

Observe that $\delta_{t,p} = 0$ if $t < 0$. Even though $\delta_{t,p}$ can be considered for any positive value of p, typically $p \geq 1$ is assumed since otherwise $\|\cdot\|_p$ is no longer a norm (we will work with $p = 4$). For completeness, we recall Definition 3 of Wu (2005).

Definition 1. The series $\{X_t\}_{t\in\mathbb{Z}}$ of the form (1) is p-strong stable if

$$\Theta_p = \sum_{t=0}^{\infty} \delta_{t,p} < \infty, \quad p \geq 1 \tag{10}$$

with the $\delta_{t,p}$ defined in Equations (9) and (8).

The decay rate of the physical dependence measure $\delta_{t,2}$ is related to the decay rate of the autocovariances $\gamma(h)$. For $r \geq 0$,

$$\sum_{h=1}^{\infty} h^r |\gamma(h)| \leq \sum_{t=0}^{\infty} \delta_{t,2} \sum_{h=1}^{\infty} h^r \delta_{h,2} \tag{11}$$

since $|\gamma(h)| \leq \sum_{t=0}^{\infty} \delta_{t,2}\delta_{t+h,2}$ (see Xiao and Wu 2012, Lemma 8) so that $\sum_{h=1}^{\infty} h^r|\gamma(h)| < \infty$ if $\sum_{h=1}^{\infty} h^r \delta_{h,2} < \infty$. In particular, $\Theta_2 < \infty$ implies absolute summability of the autocovariances. The decay rate of the physical dependence measure is also related to the summability of joint cumulants up to certain orders (see Shao and Wu 2007a, Theorem 4.1).

Wu (2005) also considers the case when instead of replacing ε_0 with its independent copy, ε_t's in Equation (1) with indices in some index set $I \subset \mathbb{Z}$ are replaced with their independent copies. If all but the first m ε_t's in Equation (1) are replaced with their independent copies, we obtain the concept of L^p-m-approximability. Again, for completeness, we recall Definition 2.1 of Hörmann and Kokoszka (2010).

Definition 2. The series $\{X_t\}_{t\in\mathbb{Z}}$ of the form (1) is L^p-m-approximable if

$$\sum_{t=1}^{\infty} \|X_t - X_t^{(t)}\|_p < \infty, \quad p \geq 1 \tag{12}$$

where

$$X_t^{(m)} = g\left(\varepsilon_t, \ldots, \varepsilon_{t-m+1}, \varepsilon'_{t-m}, \varepsilon'_{t-m-1}, \ldots\right) \tag{13}$$

Observe that $\|X_t - X_t^{(t)}\|_p = \|X_s - X_s^{(t)}\|_p$ for $s, t \in \mathbb{Z}$ so that the right-hand side of (12) can be formulated solely in terms of X_1 and the approximations $X_1^{(t)}$ with $t \geq 1$.

The following lemma shows that L^p-m-approximability implies p-strong stability.

Lemma 1. *The inequality*

$$\delta_{t,p} = \|X_t - X_{t,\{0\}}\|_p \leq 2\|X_t - X_t^{(t)}\|_p \tag{14}$$

holds for $t \geq 1$ and $p \geq 1$ with X_t, $X_{t,\{0\}}$, and $X_t^{(t)}$ given by (1), (8), and (13), respectively.

Proof of Lemma 1. The proof essentially follows from the proof of Lemma 3.0⋆ of van Delft (2019) (see page 5 therein), but we give it here for the sake of convenience (the published version, van Delft (2020), does not contain this lemma).

Denote the σ-algebras generated by the random elements $\varepsilon_t, \varepsilon_{t-1}, \ldots$ with $t \in \mathbb{Z}$ and $\varepsilon_t, \ldots, \varepsilon_s$ with $t \geq s$ by $\mathcal{F}_t = \sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$ and $\mathcal{F}_{t,s} = \sigma(\varepsilon_t, \ldots, \varepsilon_s)$, respectively. We have that

$$\|X_t - X_{t,\{0\}}\|_p \leq \|X_t - \mathrm{E}[X_t|\mathcal{F}_{t,1}]\|_p + \|\mathrm{E}[X_t|\mathcal{F}_{t,1}] - X_{t,\{0\}}\|_p$$

Since $\{\varepsilon'_t : t \in \mathbb{Z}\}$ is an independent copy of $\{\varepsilon_t : t \in \mathbb{Z}\}$,

$$\mathrm{E}[X_t|\mathcal{F}_{t,1}] = \mathrm{E}[X_t^{(t)}|\mathcal{F}_{t,1}] = \mathrm{E}[X_t^{(t)}|\mathcal{F}_t]$$

almost surely and hence

$$\begin{aligned}\|X_t - \mathrm{E}[X_t|\mathcal{F}_{t,1}]\|_p &= \|\mathrm{E}[X_t|\mathcal{F}_t] - \mathrm{E}[X_t^{(t)}|\mathcal{F}_t]\|_p \\ &\leq \|X_t - X_t^{(t)}\|_p \end{aligned} \tag{15}$$

using Jensen's inequality for conditional expectations with $p \geq 1$. We conclude the proof by noticing that $\mathrm{E}[X_t|\mathcal{F}_{t,1}]$ does not depend on ε_0 and hence

$$\|\mathrm{E}[X_t|\mathcal{F}_{t,1}] - X_{t,\{0\}}\|_p = \|\mathrm{E}[X_t|\mathcal{F}_{t,1}] - X_t\|_p$$

which is bounded in the same way as in Equation (15). The proof is complete. □

Remark 1. The L^p-m-approximability is in fact strictly stronger than p-strong stability. Consider, e.g., a linear process $\{X_t\}_{t\in\mathbb{Z}}$, where $X_t = \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}$ for $t \in \mathbb{Z}$, $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of iid random variables such that $\mathrm{E}\varepsilon_0 = 0$ and $\mathrm{E}\varepsilon_0^2 < \infty$, while $\{\psi_j\}_{j\geq 0}$ is a sequence such that $\sum_{j=0}^{\infty}\psi_j^2 < \infty$. Observe that

$$\sum_{t=0}^{\infty}\|X_t - X_{t,\{0\}}\|_2 = 2^{1/2}\|\varepsilon_0\|_2 \sum_{t=0}^{\infty}|\psi_t| \quad \text{and}$$

$$\sum_{t=1}^{\infty}\|X_t - X_t^{(t)}\|_2 = 2^{1/2}\|\varepsilon_0\|_2 \sum_{t=1}^{\infty}\left(\sum_{j=t}^{\infty}\psi_j^2\right)^{1/2}$$

Suppose that $\psi_j = (j+1)^{-d}$ for $j \geq 0$ with some $d > 1$. Then

$$\left(\sum_{j=t}^{\infty}\psi_j^2\right)^{1/2} \sim \frac{t^{1/2-d}}{(2d-1)^{1/2}} \quad \text{as} \quad t \to \infty$$

So $\{X_t\}_{t\in\mathbb{Z}}$ is 2-strong stable for any value of $d > 1$ but $\{X_t\}_{t\in\mathbb{Z}}$ is L^2-m-approximable only if $d > 3/2$.

3 | Estimation of the Long-Run Variance

Consider the time series $\{X_t\}_{t\in\mathbb{Z}}$ and its spectral density f, both defined in Section 2.1. The long-run variance of $\{X_t\}_{t\in\mathbb{Z}}$ is defined as

$$\tau_X := \sum_{k=-\infty}^{\infty} \gamma(h) = 2\pi f(0)$$

The estimation of τ_X can thus be reduced to the estimation of $f(0)$, and this approach is commonly used. A natural estimator of $f(0)$, known as the smoothed periodogram or the nonparametric spectral estimator given by (3). The chief contribution of this paper is showing that the estimator Q_n is asymptotically normal, with the rate $m^{-1/2}$, for a broad class of nonlinear moving averages of the form (1). We work under the following general assumption.

Assumption 1. The series $\{X_t\}_{t\in\mathbb{Z}}$ admits representation (1) and for some $\eta > 3$,

$$\|X_t - X_t^{(m)}\|_4 = O(m^{-\eta}) \quad \text{as} \quad m \to \infty \tag{16}$$

It follows from Lemma 1 that the bound in Assumption 1 holds for the physical dependence measure $\delta_{t,4}$. Using inequality (11) and the fact that $\delta_{t,2} \le \delta_{t,4}$, we see that Assumption 1 implies that

$$\sum_{h=1}^{\infty} h^r |\gamma(h)| < \infty, \quad 0 \le r < \eta - 1 \tag{17}$$

which in turn implies that $f(0)$ is well-defined under Assumption 1 due to the absolute summability of the autocovariances.

We now state our main result.

Theorem 1. *Assume that* $\mathrm{E}X_0 = 0$, $\mathrm{E}|X_0|^4 < \infty$, *Assumption 1 holds and* $m = m_n \to \infty$, *but* $m = o(n^{4/5})$, *as* $n \to \infty$. *Then*

$$\frac{\sqrt{m}}{f(0)}[Q_n - f(0)] = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \left(\frac{I_n(\omega_j)}{f(0)} - 1 \right) \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } n \to \infty \tag{18}$$

where Q_n *is defined in Equation (3) and* $f(0)$ *in Equation (6).*

Remark 2. The smoothed periodogram estimator typically has the form

$$\hat{f}(\omega_k) = \frac{1}{m+1} \sum_{j=-m/2}^{m/2} I_n\left(\omega_k + \frac{2\pi j}{n}\right)$$

Thus, for ω_k away from zero $\mathrm{Var}[\hat{f}(\omega_k)] \approx f(\omega_k)^2/m$, whereas at $k = 0$, $\mathrm{Var}[\hat{f}(\omega_k)] \approx 2f(\omega_k)^2/m$ because $I_n(\omega_j) = I_n(-\omega_j)$. The estimator Q_n does not include the ω_{-j}, so the factor 2 does not appear.

Theorem 1 follows from Theorems 2 and 3 formulated below. We decompose the sequence in Equation (18) into two terms

$$\frac{\sqrt{m}}{f(0)}[Q_n - f(0)] = \frac{\sqrt{m}}{f(0)}(Q_n - \mathrm{E}Q_n) + \frac{\sqrt{m}}{f(0)}(\mathrm{E}Q_n - f(0)) \tag{19}$$

Theorem 2, which implies that the first term on the right-hand side of (19) converges to $\mathcal{N}(0,1)$ in distribution, is established under much weaker assumptions than Theorem 1. In particular, the L^p-m-approximability of Assumption 1 is replaced by the weaker p-strong stability (by Lemma 1, Assumption 1 implies that $\Theta_4 < \infty$). Furthermore, only the weakest possible assumption on the rate of m is imposed in Theorem 2 (Assumption 2 below). We need to impose a stronger assumption on the growth rate of m as well as the decay rate of the covariances in Theorem 1 than in Theorem 2 to ensure that the second term on the right-hand side of (19) is asymptotically negligible. Assumption 1 implies that $\sum_{h=1}^{\infty} h^2|\gamma(h)| < \infty$ and hence, according to Theorem 3, $\mathrm{E}Q_n - f(0) = O(\max\{n^{-1}, (m/n)^2\})$, as $n \to \infty$. The assumption in Theorem 1 that $m = o(n^{4/5})$ as $n \to \infty$ then implies that $\sqrt{m}(m/n)^2 = (m/n^{4/5})^{5/2} = o(1)$, as $n \to \infty$. If $\eta > 1$ in Assumption 1, the autocovariances are absolutely summable, but this only implies that $\mathrm{E}Q_n - f(0) = o(1)$ as $n \to \infty$ (see Theorem 3 below), which is not sufficient to replace $\mathrm{E}Q_n$ with $f(0)$ as the centering sequence in the central limit theorem.

Observe that (18) implies that

$$\frac{1}{m} \sum_{j=1}^{m} \left(\frac{I_n(\omega_j)}{f(0)} - 1 \right) = O_p(m^{-1/2}), \quad \text{as } n \to \infty \tag{20}$$

Both (18) and (20) are used in theoretical work related to the estimation of the long-run variance. We give an application in Section 4.

For ease of reference, we formulate the standard assumption on the growth rate of m.

Assumption 2. As $n \to \infty$, $m \to \infty$, but $m = o(n)$.

Theorem 2. *Assume that* $\mathrm{E}X_0 = 0$, $\mathrm{E}|X_0|^4 < \infty$, $\Theta_4 < \infty$ *with* Θ_4 *defined by (10), and Assumption 2 holds. Then*

$$\frac{\sqrt{m}}{f(0)}(Q_n - \mathrm{E}Q_n) \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } n \to \infty \tag{21}$$

where Q_n *is defined in Equation (3) and* $f(0)$ *in Equation (6).*

The proof of Theorem 2 is given in Section 6, which follows from a general central limit theorem for weighted sums of periodogram ordinates, which in turn relies on a general central limit theorem for quadratic forms established by Liu and Wu (2010) (see Theorem 6 therein, Wu and Shao 2007 consider a similar estimator).

Next, we investigate the bias of the estimator Q_n. The bias of a general lag window estimator (2) depends on the curvature properties of the spectral density function and the particular lag windows. The classical results (see Theorem 9.3.3 of Anderson 1971 and Chapter 6 of Priestley 1981) establish the decay rate of the bias only under the additional assumption that the lag windows are of the scale parameter and use the concept of the characteristic exponent introduced by Parzen (1957). Since the lag windows of the estimator Q_n are not of the scale parameter form, we establish the decay rate of the bias of Q_n in the next theorem (the proof of Theorem 3 is given in Section 6).

Theorem 3. *Suppose that* $\mathrm{E}X_0 = 0$, $\mathrm{E}X_0^2 < \infty$, $\sum_{h=1}^{\infty} h^r |\gamma(h)| < \infty$ *with some* $r \geq 0$ *and Assumption 2 holds. Then*

$$\mathrm{E}Q_n - f(0) = \begin{cases} o(1) & \text{if } r = 0 \\ O((m/n)^r) & \text{if } 0 < r \leq 1 \\ O(\max\{n^{-1}, (m/n)^{\min\{2,r\}}\}) & \text{if } r > 1 \end{cases}$$

where Q_n *is defined in Equation (3) and* $f(0)$ *in Equation (6).*

Recall that the decay rate of $\delta_{t,2}$ implies a decay rate of the autocovariances. In particular, if $\sum_{h=1}^{\infty} h^r \delta_{h,2} < \infty$, then $\sum_{h=1}^{\infty} h^r |\gamma(h)| < \infty$ (see inequality (11)).

4 | Application to Change Point Analysis

A problem that has attracted some attention over the past twenty years is distinguishing between long memory, or long-range dependence (LRD), and changes in mean. Stationary time series used to model LRD exhibit spurious changes in the mean that Benoit Mandelbrot called the Joseph effect, referring to periods of famine and plenty. This is a fundamental characteristic of all stationary long memory models, see, for example, Beran et al. (2013). Berkes et al. (2006) proposed a significance test of the null hypothesis of changes in the mean against a long memory alternative. They review earlier exploratory research on this problem. Under the null hypothesis, the observations X_t satisfy

$$X_t = \begin{cases} \mu + r_t, & 1 \leq t \leq n^\star \\ \mu + \Delta + r_t, & n^\star + 1 \leq t \leq N \end{cases} \tag{22}$$

where μ is the base mean level, Δ the magnitude of change at an unknown change point $n^\star$ and $\{r_t\}_{t\in\mathbb{Z}}$ is a stationary, weakly dependent series. Berkes et al. (2006) characterized the weak dependence of the r_t by the convergence of their partial sums to the Wiener process and summability conditions on the autocovariances and fourth-order cumulants. Baek and Pipiras (2012) pointed out that the test of Berkes et al. (2006) can have low power and proposed a more powerful test of (22) based on the local Whittle estimator (LWE) of the Hurst parameter, H. However, they could justify their procedure only for *linear* processes $\{r_t\}_{t\in\mathbb{Z}}$. They relied on results of Robinson (1995) valid only for linear time series. We show in this section that the approach of Baek and Pipiras (2012) is applicable also to nonlinear processes satisfying the assumptions of Theorem 1. All results of this section are proven in Section 6.4.

We begin with the definitions of the Hurst exponent H.

Definition 3. Suppose $\{X_t\}_{t\in\mathbb{Z}}$ is a second-order stationary time series. If its spectral density satisfies

$$f(\omega) = |\omega|^{1-2H} g(\omega), \quad \text{for some } 0 < H < 1$$

where g is continuous on $[-\pi, \pi]$ with $g(\omega) > 0$ in a neighborhood of $\omega = 0$, then H is called the Hurst exponent of the series $\{X_t\}$.

Our objective is to test

H_0: Model (22) holds with a series $\{r_t\}$ satisfying Definition 3 with $H = 1/2$.

vs.

H_A: The series $\{X_t\}$ satisfies Definition 3 with $H > 1/2$.

The test is based on the following estimator. More background can be found in Chapter 8 of Giraitis et al. (2012).

Definition 4. The LWE of the Hurst exponent H is defined as

$$\hat{H} = \arg\min_{H\in\Theta} L(H) \tag{23}$$

where $\Theta = [\Delta_1, \Delta_2]$ with $0 < \Delta_1 < \Delta_2 < 1$ and

$$L(H) = \log\left(\frac{1}{m}\sum_{l=1}^{m} \omega_l^{2H-1} I_n(\omega_l)\right) - (2H-1)\frac{1}{m}\sum_{l=1}^{m} \log\omega_l \tag{24}$$

As noted above, currently, the validity of the test of Baek and Pipiras (2012) is established only if the series $\{r_t\}$ in the null hypothesis is a linear process. We show that the test retains correct asymptotic size if the $\{r_t\}$ is a Bernoulli shift satisfying Assumption 1.

To construct the test statistic, the first step is to eliminate the effect of a potential change point that can severely bias the LWE. Consider the residuals

$$R_t = \begin{cases} X_t - \frac{1}{\hat{n}}\sum_{i=1}^{\hat{n}} X_i, & 1 \leq t \leq \hat{n} \\ X_t - \frac{1}{N-\hat{n}}\sum_{i=\hat{n}+1}^{N} X_i, & \hat{n}+1 \leq t \leq N \end{cases}$$

where $\hat{n}$ is a change point estimator. We use the standard CUSUM estimator

$$\hat{n} = \min\left\{ k : \left|\sum_{j=1}^{k} X_j - \frac{k}{N}\sum_{j=1}^{N} X_j\right| = \max_{1\leq s\leq N}\left|\sum_{j=1}^{s} X_j - \frac{s}{N}\sum_{j=1}^{N} X_j\right| \right\}$$

The test statistic is

$$T^{(R)} = 2\sqrt{m}\left(\hat{H}^{(R)} - \frac{1}{2}\right) \tag{25}$$

where $\hat{H}^{(R)}$ is the LWE based on the residual process $\{R_t\}$. We will show that $T^{(R)}$ is asymptotically standard normal. The null hypothesis is rejected if $T^{(R)} > q_{1-\alpha}$, where $q_{1-\alpha}$ is the $(1-\alpha)$th quantile of the standard normal distribution.

Theorem 4. *Suppose model (22) with the* r_t *satisfying* $\mathrm{E}r_0 = 0$, $\mathrm{E}|r_0|^4 < \infty$ *and Assumption 1. In addition, assume that*

i. $n^\star = \lfloor N\theta \rfloor$ *for some* $\theta \in (0,1)$;

ii. *the change in mean level* $\Delta = \Delta(N)$ *and the change point estimator* $\hat{n}$ *satisfy*

$$N\Delta^2 \to \infty, \quad \Delta^2 |\hat{n} - n^\star| = O_P(1), \quad \text{as } N \to \infty$$

If

$$\frac{1}{m} + \frac{m^5(\log m)^2}{N^4} + \frac{m(\log m)^2}{N\Delta^2} \to 0, \quad \text{as } N \to \infty$$

then $T^{(R)} \xrightarrow{d} \mathcal{N}(0,1)$.

The proof of Theorem 4 follows the lines of the proof Theorem 3, of Baek and Pipiras (2012) as long as the following result holds and is applied to $X_t = r_t$.

Theorem 5. *Suppose* $\mathrm{E}X_0 = 0, \mathrm{E}|X_0|^4 < \infty$ *and Assumption 1 holds. If*

$$\frac{1}{m} + \frac{m^5(\log m)^2}{n^4} \to 0, \quad \text{as } n \to \infty \tag{26}$$

then

$$2\sqrt{m}\left(\hat{H} - \frac{1}{2}\right) \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } n \to \infty$$

The proof of Theorem 5 uses a similar argument as the proof of Theorem 2 of Robinson (1995). One can reduce the argument to showing that (18) in Theorem 1 holds and

$$\tilde{Q}_n := \frac{1}{\sqrt{m}} \sum_{l=1}^{m} v_{l,m} \left(\frac{I_n(\omega_l)}{f(0)} - 1 \right) \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } n \to \infty \tag{27}$$

where

$$v_{l,m} = \log l - \frac{1}{m} \sum_{j=1}^{m} \log j \tag{28}$$

The specific form of the weights $v_{l,m}$ follows from the local Whittle likelihood (24). We see that the proof of Theorem 4 can be reduced to known arguments, Theorem 1 and following result proven in Section 6.4.

Theorem 6. *Convergence (27) holds under the assumptions of Theorem 5.*

5 | A Small Simulation Study

The finite sample normality of spectral density estimators has been examined in many simulation studies, we refer to Das et al. (2021) for a study that uses a symmetrized version of Q_n, among many other estimators. We focus here on simulations related to the change point problem explained in Section 4. We examine the finite sample behavior of the test statistic $T^{(R)}$ defined in Equation (25) and the test based on it. We consider two nonlinear processes used in modeling returns on financial assets. These models are not considered by Baek and Pipiras (2012) and are not covered by their theory.

Definition 5. The series $\{r_t\}$ is said to be a GARCH(1, 1) process if it satisfies the equations

$$r_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{29}$$

where $\alpha_0 > 0$, $\alpha_1, \beta_1 \geq 0$, $\alpha_1 + \beta_1 < 1$ and the ϵ_t are iid random variables with mean zero, unit variance, and finite fourth moment.

The condition $\alpha_1 + \beta_1 < 1$ implies that the GARCH(1, 1) process has a second-order stationary solution (which is also strictly stationary) with a representation of the form (1). A necessary and sufficient condition for the existence of the fourth moments of the GARCH(1, 1) process is given by

$$\mathrm{E}|\beta_1 + \alpha_1 \epsilon_0^2|^2 < 1 \tag{30}$$

(we refer to Francq and Zakoian (2019) for an extensive review of GARCH processes). Wu and Min (2005) established (see Proposition 3 therein) that condition (30) also implies that the GARCH(1, 1) process satisfies the geometric-moment contraction (GMC) condition, that is, $\mathrm{E}|r_n - r_n^{(n)}|^4 \leq C\rho^n$, for some $C < \infty$ and $\rho \in (0,1)$, where $r_n^{(n)}$ is defined by (13) for $n \geq 1$. Hence, Assumption 1 is satisfied provided that condition (30) holds.

As the next example, we consider the simplest stochastic volatility model.

Definition 6. The stochastic volatility (SV) model is given by

$$r_t = \exp(x_t/2)\epsilon_t, \quad x_t = \alpha + \phi x_{t-1} + w_t \tag{31}$$

where $\alpha \in \mathbb{R}$, $|\phi| < 1$, $w_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_w^2)$, $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0,1)$ and $\{w_t\}$ and $\{\epsilon_t\}$ are mutually independent.

Proposition 1. *The* r_t *specified in Definition 6 satisfy Assumption 1.*

We computed the empirical density functions of the normalized LWE based on $\{r_t\}$ for both models, which are displayed in Figures 1 and 2 presented in the supporting information. These figures show that the asymptotic normality established in Theorem 5 holds well in finite samples. We also evaluated the empirical size of the test based on Theorem 4. The results reported in Tables 1 and 2 in the supporting information show that these sizes are comparable to those obtained for linear processes.

We have applied the test of Section 4 to returns and absolute returns on the S&P 500 index over a period that Includes the COVID pandemic. For any

$$m = n^p, \quad p \in \{0.60.0.65, 0.70, 0.75\} < 0.8$$

c.f. condition (26), the test accept H_0 for returns and rejects H_0 for absolute returns. It appears that the absolute returns (that model volatility) are better explained by a long memory model. This agrees with many previous studies going back to the foundational work of Ding et al. (1993); Dalla (2015) provides many related references.

6 | Proofs of the Results of Sections 3 and 4

We begin with a few auxiliary lemmas that isolate technical arguments from our main proofs.

6.1 | Auxiliary Lemmas

Lemma 2. *For $x, y \in \mathbb{R}$, $k, r \in \mathbb{Z}$, and $n \geq 1$,*

$$\sum_{t=1}^{n} \cos((t-x)\omega_k)\cos((t-y)\omega_r) = \begin{cases} \frac{n}{2}[\cos(x\omega_k - y\omega_r) \\ \quad + \cos(x\omega_k + y\omega_r)] & \text{if } k+r \in n\mathbb{Z} \text{ and } k-r \in n\mathbb{Z}; \\ \frac{n}{2}\cos(x\omega_k - y\omega_r) & \text{if } k+r \in \mathbb{Z}\setminus n\mathbb{Z} \text{ and } k-r \in n\mathbb{Z}; \\ \frac{n}{2}\cos(x\omega_k + y\omega_r) & \text{if } k+r \in n\mathbb{Z} \text{ and } k-r \in \mathbb{Z}\setminus n\mathbb{Z}; \\ 0 & \text{if } k+r \in \mathbb{Z}\setminus n\mathbb{Z} \text{ and } k-r \in \mathbb{Z}\setminus n\mathbb{Z}. \end{cases}$$

Proof. Using the fact that $\cos x = (e^{ix} + e^{-ix})/2$ for $x \in \mathbb{R}$,

$$\begin{aligned} 4\sum_{t=1}^{n} &\cos((t-x)\omega_k)\cos((t-y)\omega_r) \\ &= \sum_{t=1}^{n}(e^{i(t-x)\omega_k} + e^{-i(t-x)\omega_k})(e^{i(t-y)\omega_r} + e^{-i(t-y)\omega_r}) \\ &= e^{-i(x\omega_k + y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{(k+r)t} + e^{-i(x\omega_k - y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{(k-r)t} \\ &\quad + e^{i(x\omega_k - y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{-(k-r)t} + e^{i(x\omega_k + y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{-(k+r)t}. \end{aligned}$$

We have that

$$e^{-i(x\omega_k + y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{(k+r)t} + e^{i(x\omega_k + y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{-(k+r)t} = \begin{cases} 2n\cos(x\omega_k + y\omega_r) & \text{if } k+r \in n\mathbb{Z} \\ 0 & \text{if } k+r \notin n\mathbb{Z} \end{cases}$$

Similarly,

$$e^{-i(x\omega_k - y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{(k-r)t} + e^{i(x\omega_k - y\omega_r)}\sum_{t=1}^{n}(e^{2\pi i/n})^{-(k-r)t} = \begin{cases} 2n\cos(x\omega_k - y\omega_r) & \text{if } k-r \in n\mathbb{Z} \\ 0 & \text{if } k-r \notin n\mathbb{Z} \end{cases}$$

The proof is complete. □

Lemma 3. *For $x, y \in \mathbb{R}$, $k, r \in \mathbb{Z}$, and $n \geq 1$,*

$$\sum_{t=1}^{n} \sin((t-x)\omega_k)\sin((t-y)\omega_r) = \begin{cases} \frac{n}{2}[\cos(x\omega_k - y\omega_r) \\ \quad - \cos(x\omega_k + y\omega_r)] & \text{if } k+r \in n\mathbb{Z} \text{ and } k-r \in n\mathbb{Z} \\ \frac{n}{2}\cos(x\omega_k - y\omega_r) & \text{if } k+r \in \mathbb{Z}\setminus n\mathbb{Z} \text{ and } k-r \in n\mathbb{Z} \\ -\frac{n}{2}\cos(x\omega_k + y\omega_r) & \text{if } k+r \in n\mathbb{Z} \text{ and } k-r \in \mathbb{Z}\setminus n\mathbb{Z} \\ 0 & \text{if } k+r \in \mathbb{Z}\setminus n\mathbb{Z} \text{ and } k-r \in \mathbb{Z}\setminus n\mathbb{Z} \end{cases}$$

The proof of Lemma 3 is analogous to the proof of Lemma 2 and thus is omitted.

Lemma 4. *Let $v_{l,m} = \log l - \frac{1}{m}\sum_{j=1}^{m}\log j$. Then,*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m} v_{k,m}^2 = 1 \tag{32}$$

$$\max_{1\leq k\leq m}|v_{k,m}| = O(\log m) \tag{33}$$

Proof. Note that

$$\frac{1}{m}\sum_{k=1}^{m} v_{k,m}^2 \sim \frac{1}{m}\sum_{k=1}^{m}\left(\log\left(\frac{k}{m}\right) + 1\right)^2 \to \int_0^1 (\log x + 1)^2 dx = 1$$

Moreover, (33) follows directly from Lemma 2 in Robinson (1995). The proof is complete. □

6.2 | Central Limit Theorem for Weighted Sums of the Periodogram Ordinates

Suppose that $\kappa_m = (\kappa_{1,m}, \ldots, \kappa_{m,m}) \in \mathbb{R}^m$ with $m \geq 1$ and consider a weighted sum of the periodogram ordinates

$$W_n = \frac{1}{\|\kappa_m\|_2^2}\sum_{j=1}^{m}\kappa_{j,m} I_n(\omega_j) \tag{34}$$

where $\|\kappa_m\|_p = (|\kappa_{1,m}|^p + \cdots + |\kappa_{m,m}|^p)^{1/p}$ for $p \geq 1$ and the periodogram ordinates are defined in Equation (7). Observe that Q_n given in Equation (4) and $\tilde{Q}_n$ given in Equation (27) can be expressed as special cases of (34). The next theorem establishes a central limit theorem for the appropriately standardized weighted sum of the periodogram ordinates.

Theorem 7. *Assume that $\mathrm{E}X_0 = 0$, $\mathrm{E}|X_0|^4 < \infty$, $\Theta_4 < \infty$ with Θ_4 defined by (10), and Assumption 2 holds. Suppose that $\kappa_m = (\kappa_{1,m}, \ldots, \kappa_{m,m}) \in \mathbb{R}^m$ with $m \geq 1$ such that*

$$\frac{\|\kappa_m\|_4}{\|\kappa_m\|_2} \to 0 \quad \text{and} \quad \frac{\sum_{j=1}^{m}\kappa_{j,m}^2 \sin^2(\omega_j/2)}{\|\kappa_m\|_2^2} \to 0 \quad \text{as} \quad m \to \infty \tag{35}$$

where $\omega_j = 2\pi j/n$ are the Fourier frequencies with $1 \leq j \leq m$. Then

$$\frac{\|\kappa_m\|_2}{f(0)}(W_n - \mathrm{E}W_n) \xrightarrow{d} \mathcal{N}(0,1)$$

as $n \to \infty$, where W_n and $f(0)$ are defined in Equations (34) and (6) respectively.

The proof of Theorem 7 is based on a general central limit theorem for quadratic forms established by Liu and Wu (2010) (Thereom 6 therein), which we state here for the sake of convenience. $\mathbf{1}_A : \mathbb{R} \to \{0,1\}$ denotes the indicator function of a set $A \subset \mathbb{R}$.

Theorem 8 (Theorem 6 of Liu and Wu (2010)). *Let $a_{n,j} = b_{n,j}e^{ij\lambda}$, where $\lambda \in \mathbb{R}$, $b_{n,j} \in \mathbb{R}$ with $b_{n,j} = b_{n,-j}$, and*

$$T_n = \sum_{1\le j,j'\le n} a_{n,j-j'} X_j X_{j'} \quad \text{and}$$

$$\sigma_n^2 = (1+\mathbf{1}_{\mathbb{Z}}(\lambda/\pi))\sum_{k=1}^{n}\sum_{t=1}^{n} b_{n,t-k}^2$$

Assume that $\mathrm{E}X_0 = 0$, $\mathrm{E}|X_0|^4 < \infty$, $\Theta_4 < \infty$ *with* Θ_4 *defined by (10), and*

$$\max_{0\le t\le n} b_{n,t}^2 = o(\varsigma_n^2), \quad \text{where } \varsigma_n^2 = \sum_{t=1}^{n} b_{n,t}^2 \tag{36}$$

$$n\varsigma_n^2 = O(\sigma_n^2) \tag{37}$$

$$\sum_{k=1}^{n}\sum_{t=1}^{k-1}\left|\sum_{j=1+k}^{n} a_{n,k-j}a_{n,t-j}\right|^2 = o(\varsigma_n^4) \tag{38}$$

$$\sum_{k=1}^{n}|b_{n,k}-b_{n,k-1}|^2 = o(\varsigma_n^2) \tag{39}$$

Then for $0 \le \lambda < 2\pi$, $\sigma_n^{-1}(T_n - \mathrm{E}T_n) \xrightarrow{d} N(0, 4\pi^2 f^2(\lambda))$.

We are now ready to prove Theorem 7.

Proof of Theorem 7. We use Theorem 8 with $\lambda = 0$ to prove Theorem 7. W_n is a quadratic form given by

$$\begin{aligned} W_n &= \frac{1}{\|\kappa_m\|_2^2}\sum_{j=1}^{m}\kappa_{j,m}\cdot\frac{1}{2\pi n}\sum_{s,t=1}^{n} X_s X_t \cos((s-t)\omega_j) \\ &= \sum_{s,t=1}^{n} a_{n,s-t} X_s X_t \end{aligned}$$

where

$$a_{n,t} = \frac{1}{2\pi n\|\kappa_m\|_2^2}\sum_{j=1}^{m}\kappa_{j,m}\cos(t\omega_j)$$

for $n \ge 1$ and $|t| < n$.

For sufficiently large values of n and $k \in \mathbb{Z}$,

$$\begin{aligned} \sum_{t=1}^{n} a_{n,t-k}^2 &= \frac{1}{4\pi^2 n^2\|\kappa_m\|_2^4}\sum_{t=1}^{n}\left|\sum_{j=1}^{m}\kappa_{j,m}\cos((t-k)\omega_j)\right|^2 \\ &= \frac{1}{8\pi^2 n\|\kappa_m\|_2^2} \end{aligned} \tag{40}$$

since

$$\begin{aligned} \sum_{t=1}^{n}\left|\sum_{j=1}^{m}\kappa_{j,m}\cos((t-k)\omega_j)\right|^2 &= \sum_{j=1}^{m}\sum_{l=1}^{m}\kappa_{j,m}\kappa_{l,m} \\ &\quad\times\sum_{t=1}^{n}\cos((t-k)\omega_j)\cos((t-k)\omega_l) \\ &= \frac{n}{2}\|\kappa_m\|_2^2 \end{aligned}$$

using Lemma 2 as $j + l \in \mathbb{Z}\setminus n\mathbb{Z}$ for sufficiently large values of n due to Assumption 2.

Using (40),

$$\sigma_n^2 = 2\sum_{k=1}^{n}\sum_{t=1}^{n} a_{n,t-k}^2 = \frac{1}{4\pi^2\|\kappa_m\|_2^2} \quad \text{and}$$

$$\varsigma_n^2 = \sum_{t=1}^{n} a_{n,t}^2 = \frac{1}{8\pi^2 n\|\kappa_m\|_2^2}$$

for sufficiently large values of n.

Using the Cauchy-Schwarz inequality and the fact that $|\cos x| \le 1$ for $x \in \mathbb{R}$,

$$\begin{aligned} \max_{0\le t\le n} a_{n,t}^2 &= \frac{1}{4\pi^2 n^2\|\kappa_m\|_2^4}\max_{0\le t\le n}\left|\sum_{j=1}^{m}\kappa_{j,m}\cos(t\omega_j)\right|^2 \\ &\le \frac{1}{4\pi^2 n^2\|\kappa_m\|_2^4}\|\kappa_m\|_2^2\max_{0\le t\le n}\left\{\sum_{j=1}^{m}\cos^2(t\omega_j)\right\} \\ &\le \frac{m}{4\pi^2 n^2\|\kappa_m\|_2^2} \end{aligned}$$

and hence

$$\frac{\max_{0\le t\le n} a_{n,t}^2}{\varsigma_n^2} = \frac{m}{4\pi^2 n^2\|\kappa_m\|_2^2}\cdot 8\pi^2 n\|\kappa_m\|_2^2 = \frac{2m}{n} = o(1) \quad \text{as} \quad n \to \infty$$

due to Assumption 2 so that (36) is satisfied. (37) is also satisfied since

$$\frac{n\varsigma_n^2}{\sigma_n^2} = \frac{4\pi^2\|\kappa_m\|_2^2}{8\pi^2\|\kappa_m\|_2^2} = \frac{1}{2}$$

for sufficiently large values of n.

Next, we verify that (38) holds. We have that

$$\begin{aligned} &\sum_{k=1}^{n}\sum_{t=1}^{k-1}\left|\sum_{j=k+1}^{n} a_{n,k-j}a_{n,t-j}\right|^2 = \\ &\le \sum_{k=1}^{n}\sum_{t=1}^{n}\left[\sum_{j=1}^{n} a_{n,k-j}^2 a_{n,t-j}^2 + 2\right. \\ &\quad\left.\times\sum_{j=k+1}^{n-1}\sum_{l=j+1}^{n} a_{n,k-j}a_{n,t-j}a_{n,k-l}a_{n,t-l}\right] \\ &= \left[\sum_{j=1}^{n}\sum_{k=1}^{n} a_{n,k-j}^2\sum_{t=1}^{n} a_{n,t-j}^2 + 2\sum_{k=1}^{n}\sum_{j=k+1}^{n-1} a_{n,k-j}\right. \\ &\quad\left.\times\sum_{l=j+1}^{n} a_{n,k-l}\left(\sum_{t=1}^{n} a_{n,t-j}a_{n,t-l}\right)\right] \\ &= \left[\sum_{j=1}^{n}\left|\sum_{t=1}^{n} a_{n,t-j}^2\right|^2 + 2\sum_{k=1}^{n}\sum_{j=k+1}^{n-1} a_{n,k-j}\beta_{n,k,j}\right] \end{aligned}$$

where

$$\beta_{n,k,j} := \sum_{l=j+1}^{n} a_{n,k-l}\alpha_{n,j,l} \quad \text{and} \quad \alpha_{n,j,l} := \sum_{t=1}^{n} a_{n,t-j}a_{n,t-l}$$

By the Cauchy-Schwarz inequality,

$$\left|2\sum_{k=1}^{n}\sum_{j=k+1}^{n-1} a_{n,k-j}\beta_{n,k,j}\right| \le 2\sum_{k=1}^{n}\left|\sum_{j=k+1}^{n-1} a_{n,k-j}\beta_{n,k,j}\right|$$

$$\leq 2\sum_{k=1}^{n}\left(\sum_{j=k+1}^{n-1}a_{n,k-j}^2\right)^{1/2}\left(\sum_{j=k+1}^{n-1}\beta_{n,k,j}^2\right)^{1/2}$$
$$\leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{n}a_{n,k-j}^2\right)^{1/2}\left(\sum_{j=1}^{n}\beta_{n,k,j}^2\right)^{1/2}$$

and

$$\beta_{n,k,j}^2 = \left|\sum_{l=j+1}^{n} a_{n,k-l}\alpha_{n,j,l}\right|^2 \leq \left(\sum_{l=j+1}^{n} a_{n,k-l}^2\right)\left(\sum_{l=j+1}^{n}\alpha_{n,j,l}^2\right) \leq \left(\sum_{l=1}^{n}a_{n,k-l}^2\right)\left(\sum_{l=1}^{n}\alpha_{n,j,l}^2\right) \tag{41}$$

Using Lemma 2 twice,

$$\begin{aligned}\sum_{l=1}^{n}\alpha_{n,j,l}^2 &= \sum_{l=1}^{n}\left|\sum_{t=1}^{n}\frac{1}{2\pi n\|\kappa_m\|_2^2}\times\sum_{r=1}^{m}\kappa_{r,m}\cos((t-j)\omega_r)\frac{1}{2\pi n\|\kappa_m\|_2^2}\times\sum_{s=1}^{m}\kappa_{s,m}\cos((t-l)\omega_s)\right|^2\\
&= \frac{1}{16\pi^4 n^4\|\kappa_m\|_2^8}\sum_{l=1}^{n}\left|\sum_{r=1}^{m}\sum_{s=1}^{m}\kappa_{r,m}\kappa_{s,m}\times\sum_{t=1}^{n}\cos((t-j)\omega_r)\cos((t-l)\omega_s)\right|^2\\
&= \frac{1}{64\pi^4 n^2\|\kappa_m\|_2^8}\sum_{l=1}^{n}\left|\sum_{r=1}^{m}\kappa_{r,m}^2\cos((j-l)\omega_r)\right|^2\\
&= \frac{1}{64\pi^4 n^2\|\kappa_m\|_2^8}\sum_{l=1}^{n}\sum_{r=1}^{m}\sum_{s=1}^{m}\kappa_{r,m}^2\cos((j-l)\omega_r)\kappa_{s,m}^2\cos((j-l)\omega_s)\\
&= \frac{1}{64\pi^4 n^2\|\kappa_m\|_2^8}\sum_{r=1}^{m}\sum_{s=1}^{m}\kappa_{r,m}^2\kappa_{s,m}^2\sum_{l=1}^{n}\cos((j-l)\omega_r)\cos((j-l)\omega_s)\\
&= \frac{1}{64\pi^4 n^2\|\kappa_m\|_2^8}\|\kappa_m\|_4^4\cdot\frac{n}{2}\\
&= \frac{\|\kappa_m\|_4^4}{128\pi^4 n\|\kappa_m\|_2^8}\end{aligned} \tag{42}$$

for sufficiently large values of n.

Using (41), (40), and (42),

$$\beta_{n,k,j}^2 \leq \left(\sum_{l=1}^{n}a_{n,k-l}^2\right)\left(\sum_{l=1}^{n}\alpha_{n,j,l}^2\right) \leq \frac{1}{8\pi^2 n\|\kappa_m\|_2^2}\cdot\frac{\|\kappa_m\|_4^4}{128\pi^4 n\|\kappa_m\|_2^8} = \frac{\|\kappa_m\|_4^4}{2^{10}\pi^6 n^2\|\kappa_m\|_2^{10}}$$

Using (40) again,

$$\sum_{j=1}^{n}\left|\sum_{t=1}^{n}a_{n,t-j}^2\right|^2 = \sum_{j=1}^{n}\left|\frac{1}{8\pi^2 n\|\kappa_m\|_2^2}\right|^2 = \frac{1}{64\pi^4 n\|\kappa_m\|_2^4}$$

and

$$\begin{aligned}\left|2\sum_{k=1}^{n}\sum_{j=k+1}^{n-1}a_{n,k-j}\beta_{n,k,j}\right| &\leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{n}a_{n,k-j}^2\right)^{1/2}\left(\sum_{j=1}^{n}\beta_{n,k,j}^2\right)^{1/2}\\
&\leq 2\sum_{k=1}^{n}\left(\frac{1}{8\pi^2 n\|\kappa_m\|_2^2}\right)^{1/2}\cdot\frac{n^{1/2}\|\kappa_m\|_4^2}{2^5\pi^3 n\|\kappa_m\|_2^5}\\
&= 2n\frac{1}{2^{3/2}\pi n^{1/2}\|\kappa_m\|_2}\cdot\frac{n^{1/2}\|\kappa_m\|_4^2}{2^5\pi^3 n\|\kappa_m\|_2^5}\\
&= \frac{\|\kappa_m\|_4^2}{2^{11/2}\pi^4\|\kappa_m\|_2^6}\end{aligned}$$

so that

$$\begin{aligned}\frac{\sum_{k=1}^{n}\sum_{t=1}^{k-1}|\sum_{j=k+1}^{n}a_{n,k-j}a_{n,t-j}|^2}{\sigma_n^4} &=\\
&= \sum_{k=1}^{n}\sum_{t=1}^{k-1}\left|\sum_{j=k+1}^{n}a_{n,k-j}a_{n,t-j}\right|^2\cdot 16\pi^4\|\kappa_m\|_2^4\\
&\leq \left[\frac{1}{64\pi^4 n\|\kappa_m\|_2^4}+\frac{\|\kappa_m\|_4^2}{2^{11/2}\pi^4\|\kappa_m\|_2^6}\right]\cdot 16\pi^4\|\kappa_m\|_2^4\\
&= \frac{1}{4n}+2^{-3/2}\left(\frac{\|\kappa_m\|_4}{\|\kappa_m\|_2}\right)^2\\
&= o(1)\end{aligned}$$

as $n\to\infty$ due to (35) and hence (38) holds.

We finally verify that (39) holds. Using the identity

$$\cos\theta-\cos(\varphi) = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

for $\theta,\varphi\in\mathbb{R}$ and Lemma 3,

$$\begin{aligned}\sum_{k=1}^{n}|a_{n,k}-a_{n,k-1}|^2 &=\\
&= \frac{1}{4\pi^2 n^2\|\kappa_m\|_2^4}\sum_{k=1}^{n}\left|\sum_{j=1}^{m}\kappa_{j,m}[\cos(k\omega_j)-\cos((k-1)\omega_j)]\right|^2\\
&= \frac{1}{\pi^2 n^2\|\kappa_m\|_2^4}\sum_{k=1}^{n}\left|\sum_{j=1}^{m}\kappa_{j,m}[\sin((2k-1)\omega_j/2)\sin(\omega_j/2)]\right|^2\\
&= \frac{1}{\pi^2 n^2\|\kappa_m\|_2^4}\sum_{j,l=1}^{m}\kappa_{j,m}\kappa_{l,m}\sin(\omega_j/2)\sin(\omega_l/2)\left\{\sum_{k=1}^{n}\sin((k-1/2)\omega_j)\sin((k-1/2)\omega_l)\right\}\\
&= \frac{1}{2\pi^2 n\|\kappa_m\|_2^4}\sum_{j=1}^{m}\kappa_{j,m}^2\sin^2(\omega_j/2)\end{aligned}$$

for sufficiently large values of n. Hence,

$$\begin{aligned}
&\frac{\sum_{k=1}^{n}|a_{n,k}-a_{n,k-1}|^2}{\varsigma_n^2}\\
&=\frac{1}{2\pi^2 n\|\kappa_m\|_2^4}\sum_{j=1}^{m}\kappa_{j,m}^2\sin^2(\omega_j/2)\cdot 8\pi^2 n\|\kappa_m\|_2^2\\
&=\frac{4\sum_{j=1}^{m}\kappa_{j,m}^2\sin^2(\omega_j/2)}{\|\kappa_m\|_2^2}\\
&=o(1)
\end{aligned}$$

as $n\to\infty$ due to (35) and hence (39) holds. Theorem 8 then implies that

$$\sigma_n^{-1}(W_n-\mathrm{E}W_n)\xrightarrow{d}N(0,4\pi^2f^2(0))\quad\text{as}\quad n\to\infty,$$

which completes the proof. □

6.3 | Proofs of the Theorems of Section 3

Proof of Theorem 2. We set $\kappa_{j,m}=1$ in Theorem 7 for $j=1,\ldots,m$ and $m\geq 1$. The first condition of (35) holds. Using the inequality $|\sin x|\leq|x|$ for $x\in\mathbb{R}$,

$$\frac{\sum_{j=1}^{m}\sin^2(\omega_j/2)}{m}\leq\frac{\pi^2\sum_{j=1}^{m}j^2}{n^2m}=O\left(\left(\frac{m}{n}\right)^2\right)$$

as $n\to\infty$ and the second condition of (35) also holds under Assumption 2. The proof is complete. □

Proof of Theorem 3. Observe that Q_n can be expressed in the following way

$$Q_n=\frac{1}{2\pi}\sum_{h=-(n-1)}^{n-1}a_{n,h}\hat{\gamma}(h) \tag{43}$$

for $n\geq 1$, where

$$a_{n,h}=\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)$$

for $n\geq 1$ and $|h|<n$ while $\hat{\gamma}(h)$ are sample autocovariances given by

$$\hat{\gamma}(h)=\frac{1}{n}\sum_{j=1}^{n-h}X_{j+h}X_j$$

for $0\leq h<n$ and $\hat{\gamma}(h)=\hat{\gamma}(-h)$ for $-n<h\leq 0$. Hence, we have that

$$\begin{aligned}
&\mathrm{E}\left[\frac{1}{m}\sum_{j=1}^{m}I_n(\omega_j)\right]-f(0)\\
&=\frac{1}{2\pi}\sum_{h=-(n-1)}^{n-1}\left(1-\frac{|h|}{n}\right)\gamma(h)\left[\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)\right]\\
&\quad-\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma(h)\\
&=\frac{1}{2\pi}\left[\sum_{h=-(n-1)}^{n-1}\left\{\left(1-\frac{|h|}{n}\right)\left[\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)\right]-1\right\}\right.\\
&\quad\left.\gamma(h)-\sum_{|h|\geq n}\gamma(h)\right]\\
&=\frac{1}{\pi}\left[\sum_{h=1}^{n-1}\left\{\left[\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)\right]-\frac{h}{n}\left[\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)\right]-1\right\}\right.\\
&\quad\left.\gamma(h)-\sum_{h=n}^{\infty}\gamma(h)\right]\\
&=\frac{1}{\pi}\left[\sum_{h=1}^{n-1}\left\{\left[\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)\right]-1\right.\right.\\
&\quad\left.-\frac{h}{n}\left[\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)\right]\right\}\gamma(h)\\
&\quad\left.-\sum_{h=n}^{\infty}\gamma(h)\right].
\end{aligned}$$

Split the sum into two parts

$$\begin{aligned}
&\sum_{h=1}^{n-1}\left|\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)-1\right||\gamma(h)|\\
&\quad=\sum_{h=1}^{\tau_{n,m}-1}\left|\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)-1\right||\gamma(h)|\\
&\qquad+\sum_{h=\tau_{n,m}}^{n-1}\left|\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)-1\right||\gamma(h)|
\end{aligned}\tag{44}$$

where

$$\tau_{n,m}=\left\lfloor\frac{\sqrt{3}n}{\pi\sqrt{(m+1)(2m+1)}}\right\rfloor.$$

Using the mean value theorem and the fact that $|\sin x|\leq|x|$ for $x\in\mathbb{R}$,

$$\left|\cos\left(h\cdot\frac{2\pi j}{n}\right)-\cos(0)\right|=|\sin c|\left|\frac{2\pi hj}{n}\right|\leq\left|\frac{2\pi hj}{n}\right|^2$$

with some $c\in(0,2\pi hj/n)$. Hence,

$$\begin{aligned}
&\sum_{h=1}^{\tau_{n,m}-1}\left|\frac{1}{m}\sum_{j=1}^{m}\cos(h\omega_j)-1\right||\gamma(h)|\\
&\quad\leq\sum_{h=1}^{\tau_{n,m}-1}\frac{2}{3}\left|\frac{\pi h}{n}\right|^2(m+1)(2m+1)|\gamma(h)|\\
&\quad=\frac{2}{3}\left|\frac{\pi}{n}\right|^2(m+1)(2m+1)\sum_{h=1}^{\tau_{n,m}-1}h^2|\gamma(h)|
\end{aligned}\tag{45}$$

If $0\leq r<2$,

$$\sum_{h=1}^{\tau_{n,m}-1}h^2|\gamma(h)|=\sum_{h=1}^{\tau_{n,m}-1}h^{2-r}h^r|\gamma(h)|\leq(\tau_{n,m}-1)^{2-r}\sum_{h=1}^{\tau_{n,m}-1}h^r|\gamma(h)|$$

and (45) is $O((m/n)^r)$ as $n\to\infty$. If $r\geq 2$, (45) is $O((m/n)^2)$ as $n\to\infty$. We bound the second term of (44) in the following way

$$\sum_{h=\tau_{n,m}}^{n-1} \left| \frac{1}{m} \sum_{j=1}^{m} \cos(h\omega_j) - 1 \right| |\gamma(h)| \le 2 \sum_{h=\tau_{n,m}}^{n-1} |\gamma(h)|$$
$$= 2 \sum_{h=\tau_{n,m}}^{n-1} h^{-r} h^r |\gamma(h)|$$
$$\le 2\tau_{n,m}^{-r} \sum_{h=\tau_{n,m}}^{n-1} h^r |\gamma(h)|$$
$$= o((m/n)^r)$$

as $n \to \infty$. We have that

$$\sum_{h=1}^{n-1} \frac{h}{n} \left| \frac{1}{m} \sum_{j=1}^{m} \cos(h\omega_j) \right| |\gamma(h)| \le \frac{1}{n} \sum_{h=1}^{n-1} h|\gamma(h)|$$

If $0 < r < 1$, then

$$\frac{1}{n} \sum_{h=1}^{n-1} h|\gamma(h)| = \frac{1}{n} \sum_{h=1}^{n-1} h^{1-r} h^r |\gamma(h)|$$
$$\le \frac{(n-1)^{1-r}}{n} \sum_{h=1}^{n-1} h^r |\gamma(h)| = O(n^{-r})$$

as $n \to \infty$. If $r \ge 1$, then

$$\frac{1}{n} \sum_{h=1}^{n-1} h|\gamma(h)| = O(n^{-1})$$

as $n \to \infty$. Finally,

$$\sum_{h=n}^{\infty} |\gamma(h)| = \sum_{h=n}^{\infty} h^r h^{-r} |\gamma(h)| \le n^{-r} \sum_{h=n}^{\infty} h^r |\gamma(h)| = o(n^{-r})$$

as $n \to \infty$. The proof is complete. □

6.4 | Proofs of the Results of Section 4

Proof of Theorem 6. We set $\kappa_{j,m} = \nu_{j,m}$ in Theorem 7 with $\nu_{j,m}$ given by (28) for $j = 1, \ldots, m$ and $m \ge 1$. Using Lemma 4 and the inequality $|\sin x| \le |x|$ for $x \in \mathbb{R}$,

$$\frac{\|\nu_m\|_4}{\|\nu_m\|_2} \le \frac{m^{1/4} \max_{1\le j\le m} |\nu_{j,m}|}{m^{1/2} \cdot m^{-1/2} \|\nu_m\|_2} = O\left(\frac{\log m}{m^{1/4}}\right)$$

and

$$\frac{\sum_{j=1}^{m} \nu_{j,m}^2 \sin^2(\omega_j/2)}{\|\nu_m\|_2^2} \le \frac{\pi \max_{1\le j\le m} \nu_{j,m}^2 \cdot \sum_{j=1}^{m} j^2}{n^2 \|\nu_m\|_2^2} = O\left(\frac{m^2 \log^2 m}{n^2}\right)$$

as $n \to \infty$ so that conditions (35) are satisfied under assumption (26). Theorem 7 then implies that

$$\frac{\|\nu_m\|_2}{f(0)} \left[\frac{1}{\|\nu_n\|_2^2} \sum_{j=1}^{m} \nu_{j,m} I_n(\omega_j) - \mathrm{E}\left[\frac{1}{\|\nu_n\|_2^2} \sum_{j=1}^{m} \nu_{j,m} I_n(\omega_j) \right] \right] \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty$$

To complete the proof, we show that

$$E\left[\frac{1}{\sqrt{m}} \sum_{j=1}^{m} \nu_{j,m} I_n(\omega_j) \right] = o(1) \quad \text{as} \quad n \to \infty$$

We have that

$$\mathrm{E}\left[\frac{1}{\sqrt{m}} \sum_{l=1}^{m} \nu_{l,m} I_n(\omega_l) \right] = \frac{1}{\sqrt{m}} \sum_{l=1}^{m} \nu_{l,m} \mathrm{E} I_n(\omega_l)$$
$$= \frac{1}{2\pi} \sum_{h=-(n-1)}^{n-1} \left[\frac{1}{\sqrt{m}} \sum_{l=1}^{m} \nu_{l,m} \cos(h\omega_l) \right] \times \left(1 - \frac{|h|}{n}\right) \gamma(h)$$
$$= \frac{1}{\pi} \sum_{h=1}^{n-1} \left[\frac{1}{\sqrt{m}} \sum_{l=1}^{m} \nu_{l,m} \{\cos(h\omega_l) - 1\} \right] \times \left(1 - \frac{h}{n}\right) \gamma(h)$$

since $\sum_{j=1}^{m} \nu_{j,m} = 0$ for $m \ge 1$. Using Lemma 4, the identity $1 - \cos x = 2\sin^2(x/2)$ for $x \in \mathbb{R}$, and the inequality $|\sin x| \le |x|$ for $x \in \mathbb{R}$,

$$\left| \mathrm{E}\left[\frac{1}{\sqrt{m}} \sum_{l=1}^{m} \nu_{l,m} I_n(\omega_l) \right] \right|$$
$$\le \frac{1}{\pi} \sum_{h=1}^{n-1} \frac{1}{\sqrt{m}} \sum_{l=1}^{m} |\nu_{l,m}| |\cos(h\omega_l) - 1| |\gamma(h)|$$
$$\le \frac{2C \log m}{\pi} \sum_{h=1}^{n-1} \frac{1}{\sqrt{m}} \sum_{l=1}^{m} \left(\frac{h\omega_l}{2}\right)^2 |\gamma(h)|$$
$$= \frac{2\pi C}{3} \sum_{h=1}^{n-1} h^2 |\gamma(h)| \cdot \frac{\log m}{n^2 \sqrt{m}} \sum_{l=1}^{m} l^2$$
$$= O\left(\frac{m^{5/2} \log m}{n^2}\right)$$

as $n \to \infty$ since $\sum_{h=1}^{\infty} h^2 |\gamma(h)| < \infty$ under Assumption 1 (see inequality (17)). The proof is complete. □

Proof of Proposition 1. Let $\mu = E[x_t] = \frac{\alpha}{1-\phi}$, then $x_t = \mu + \sum_{j=0}^{\infty} \phi^j w_{t-j}$. Define

$$u_t = \exp\left(\frac{x_t}{2}\right) = \exp\left(\frac{1}{2}\mu + \frac{1}{2} \sum_{j=0}^{\infty} \phi^j w_{t-j}\right)$$
$$= \exp\left(\zeta_{m-1,t} + \xi_{m,t}\right)$$

where

$$\zeta_{m-1,t} = \frac{1}{2}\mu + \frac{1}{2} \sum_{j=0}^{m-1} \phi^j w_{t-j}, \quad \xi_{m,t} = \frac{1}{2} \sum_{j=m}^{\infty} \phi^j w_{t-j}$$

Clearly, the mapping from $(w_t, w_{t-1}, \ldots)$ to u_t is measurable. For each $m \in \mathbb{N}^+$, we can approximate the u_t by

$$u_t^{(m)} = \exp\left(\zeta_{m-1,t} + \xi'_{m,t}\right), \tag{46}$$

where

$$\xi'_{m,t} = \frac{1}{2}\sum_{j=m}^{\infty}\phi^j w'_{t-j}$$

and $\{w'_{t-l},\ l \geq m, m \in \mathbb{N}^+\}$ are iid copies of w_0.

Using the identity $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$, we have

$$\begin{aligned}\left|u_t - u_t^{(m)}\right|^4 &= \exp(4\zeta_{m-1,t}) \cdot \left[\exp(\xi_{m,t}) - \exp\left(\xi'_{m,t}\right)\right]^4 \\ &= \exp(4\zeta_{m-1,t}) \cdot \left\{\exp(4\xi_{m,t}) - 4\exp\left(3\xi_{m,t} + \xi'_{m,t}\right)\right. \\ &\quad + 6\exp\left(2\xi_{m,t} + 2\xi'_{m,t}\right) - 4\exp\left(\xi_{m,t} + 3\xi'_{m,t}\right) \\ &\quad \left. + \exp\left(4\xi'_{m,t}\right)\right\}\end{aligned}$$

Let $\varphi(s)$ be the characteristic function of w_0. Then, the characteristic function of the $\xi_{m,t}$ is given by

$$\begin{aligned}\varphi_\xi(s) &= E[\exp(is\xi_{m,t})] = \prod_{j=m}^{\infty}\varphi(s\phi^j) \\ &= \exp\left\{-\frac{1}{2}\left(\frac{\phi^{2m}}{1-\phi^2}\sigma_w^2\right)s^2\right\}\end{aligned}$$

Hence,

$$\xi_{m,t} \overset{d}{=} \xi'_{m,t} \sim \mathcal{N}\left(0, \frac{\phi^{2m}}{4(1-\phi^2)}\sigma_w^2\right)$$

On the other hand,

$$\zeta_{m-1,t} \sim \mathcal{N}\left(\frac{1}{2}\mu, \frac{1-\phi^{2m}}{4(1-\phi^2)}\sigma_w^2\right)$$

We know that $E[e^{\mu+\sigma Z}] = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$ if $Z \sim \mathcal{N}(0,1)$. Consequently,

$$\begin{aligned}E\left|u_t - u_t^{(m)}\right|^4 &= \exp\left(2\mu + \frac{2(1-\phi^{2m})}{1-\phi^2}\sigma_w^2\right) \cdot \left\{2\exp\left(\frac{2\phi^{2m}}{1-\phi^2}\sigma_w^2\right)\right. \\ &\quad \left. -8\exp\left(\frac{5\phi^{2m}}{2(1-\phi^2)}\sigma_w^2\right) + 6\exp\left(\frac{2\phi^{2m}}{1-\phi^2}\sigma_w^2\right)\right\} \\ &= \exp\left(2\mu + \frac{2\sigma_w^2}{1-\phi^2}\right) \\ &\quad \cdot \left\{8 - 8\exp\left(\frac{\phi^{2m}}{2(1-\phi^2)}\sigma_w^2\right)\right\} \\ &= O(\phi^{2m})\end{aligned} \tag{47}$$

where the last equality comes from the Taylor expansion.

Finally, let $\boldsymbol{\eta}_t = (w_t, \epsilon_t)^\intercal$. Since $r_t = u_t\epsilon_t$, the mapping from $(\boldsymbol{\eta}_t, \boldsymbol{\eta}_{t-1}, \ldots)$ to r_t is measurable. For each $m \in \mathbb{N}^+$, the r_t can be approximated by $r_t^{(m)} = u_t^{(m)}\epsilon_t$, where the $u_t^{(m)}$ is given by (46). Since $\{\epsilon_t\}$ and $\{w_t\}$ are mutually independent,

$$\begin{aligned}\|r_t - r_t^{(m)}\|_4 &= \|\epsilon_0\|_4 \cdot \|u_t - u_t^{(m)}\|_4 \\ &= O\left(\phi^{m/2}\right) \\ &= O(m^{-\kappa^*})\end{aligned}$$

for any $\kappa^* > 0$. The proof is complete. □

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The authors have nothing to report.

References

Anderson, T. W. 1971. *The Statistical Analysis of Time Series*, Wiley Series in Probability and Statistics. John Wiley & Sons, Inc.

Aue, A., S. Hörmann, L. Horváth, and M. Reimherr. 2009. "Break Detection in the Covariance Structure of Multivariate Time Series Models." *Annals of Statistics* 37: 4046–4087.

Baek, C., M.-C. Düker, and V. Pipiras. 2023. "Local Whittle Estimation of High-Dimensional Long-Run Variance and Precision Matrices." *Annals of Statistics* 51, no. 6: 2386–2414.

Baek, C., P. Kokoszka, and X. Meng. 2024. "Test of Change Point Versus Long-Range Dependence in Functional Time Series." *Journal of Time Series Analysis* 45: 497–512.

Baek, C., and V. Pipiras. 2012. "Statistical Tests for a Single Change in Mean Against Long-Range Dependence." *Journal of Time Series Analysis* 33: 131–151.

Bardsley, P., L. Horváth, P. Kokoszka, and G. Young. 2017. "Change Point Tests in Functional Factor Models With Application to Yield Curves." *Econometrics Journal* 20: 86–117.

Beran, J., Y. Feng, S. Ghosh, and R. Kulik. 2013. *Long Memory Processes*. Princeton University Press.

Berkes, I., L. Horváth, P. Kokoszka, and Q.-M. Shao. 2006. "On Discriminating Between Long–Range Dependence and Changes in Mean." *Annals of Statistics* 34: 1140–1165.

Berkes, I., L. Horváth, and G. Rice. 2013. "Weak Invariance Principles for Sums of Dependent Random Functions." *Stochastic Processes and their Applications* 123: 385–403.

Berkes, I., L. Horváth, and G. Rice. 2016. "On the Asymptotic Normality of Kernel Estimators of the Long Run Covariance of Functional Time Series." *Journal of Multivariate Analysis* 144: 150–175.

Brillinger, D. R. 1969. "Asymptotic Properties of Spectral Estimates of Second Order." *Biometrika* 56: 375–390.

Chan, K. W. 2022. "Optimal Difference-Based Variance Estimators in Time Series: A General Framework." *Annals of Statistics* 50, no. 3: 1376–1400.

Chanda, K. C. 2005. "Large Sample Properties of Spectral Estimators for a Class of Stationary Nonlinear Processes." *Journal of Time Series Analysis* 26: 1–16.

Dalla, V. 2015. "Power Transformations of Absolute Returns and Long Memory Estimation." *Journal of Empirical Finance* 33: 1–18.

Das, S., S. Subba Rao, and J. Yang. 2021. "Spectral Methods for Small Sample Time Series: A Complete Periodogram Approach." *Journal of Time Series Analysis* 42: 597–621.

Dette, H., K. Kokot, and S. Volgushev. 2020. "Testing Relevant Hypotheses in Functional Time Series via Self-Normalization." *Journal of the Royal Statistical Society, Series B: Statistical Methodology* 82: 629–660.

Ding, Z., C. Granger, and R. Engle. 1993. "A Long Memory Property of Stock Market Returns and a New Model." *Journal of Empirical Finance* 1: 83–106.

Francq, C., and J.-M. Zakoian. 2019. *GARCH Models: Structure, Statistical Inference and Financial Applications*. 2nd ed. John Wiley & Sons Ltd.

Giraitis, L., H. Koul, and D. Surgailis. 2012. *Large Sample Inference for Long Memory Processes*. Imperial College Press.

Hamilton, J. D. 1994. *Time Series Analysis*. Princeton University Press.

Hörmann, S., and P. Kokoszka. 2010. "Weakly Dependent Functional Data." *Annals of Statistics* 38, no. 3: 1845–1884.

Horváth, L., P. Kokoszka, and S. Lu. 2024. "Variable Selection Based Testing for Parameter Changes in Regression With Autoregressive Dependence." *Journal of Business and Economic Statistics* 42: 1331–1343.

Horváth, L., P. Kokoszka, and R. Reeder. 2013. "Estimation of the Mean of Functional Time Series and a Two Sample Problem." *Journal of the Royal Statistical Society. Series B, Statistical Methodology* 75: 103–122.

Horváth, L., P. Kokoszka, and G. Rice. 2014. "Testing Stationarity of Functional Time Series." *Journal of Econometrics* 179: 66–82.

Kokoszka, P., H. Miao, and X. Zhang. 2015. "Functional Dynamic Factor Model for Intraday Price Curves." *Journal of Financial Econometrics* 13: 456–477.

Li, D., P. Robinson, and H. L. Shang. 2021. "Local Whittle Estimation of Long-Range Dependence for Functional Time Series." *Journal of Time Series Analysis* 42: 685–695.

Liu, W., and W. B. Wu. 2010. "Asymptotics of Spectral Density Estimates." *Econometric Theory* 26: 1218–1245.

Liu, W., H. Xiao, and W. Wu. 2013. "Probability and Moment Inequalities Under Dependence." *Statistica Sinica* 23: 1257–1272.

McElroy, T., and D. Politis. 2024. "Estimating the Spectral Density at Frequencies Near Zero." *Journal of the American Statistical Association* 119: 612–624.

Parzen, E. 1957. "On Consistent Estimates of the Spectrum of a Stationary Time Series." *Annals of Mathematical Statistics* 28, no. 2: 329–348.

Phillips, P., and K. Shimotsu. 2004. "Local Whittle Estimation in Nonstationary and Unit Root Cases." *Annals of Statistics* 32: 656–692.

Priestley, M. B. 1981. *Spectral Analysis and Time Series*. Academic Press.

Robinson, P. 1995. "Gaussian Semiparametric Estimation of Long Range Dependence." *Annals of Statistics* 23: 1630–1661.

Robinson, P. 2008. "Multiple Local Whittle Estimation in Stationary Systems." *Annals of Statistics* 36: 2508–2530.

Rosenblatt, M. 1984. "Asymptotic Normality, Strong Mixing and Spectral Density Estimates." *Annals of Probability* 12: 1167–1180.

Shao, X., and W. B. Wu. 2007a. "Local Whittle Estimation of Fractional Integration for Nonlinear Processes." *Econometric Theory* 23: 899–929.

Shao, X., and W. B. Wu. 2007b. "Asymptotic Spectral Theory for Nonlinear Time Series." *Annals of Statistics* 35: 1773–1801.

van Delft, A. 2019. "A Note on Quadratic Forms of Stationary Functional Time Series Under Mild Conditions." arXiv e-prints.

van Delft, A. 2020. "A Note on Quadratic Forms of Stationary Functional Time Series Under Mild Conditions." *Stochastic Processes and their Applications* 130: 4206–4251.

Velasco, C. 1999. "Gaussian Semiparametric Estimation of Non-Stationary Time Series." *Journal of Time Series Analysis* 20: 87–127.

Wu, W. B. 2005. "Nonlinear System Theory: Another Look at Dependence." *Proceedings of the National Academy of Sciences* 102, no. 40: 14150–14154.

Wu, W. B., and J. Mielniczuk. 2010. "A New Look at Measuring Dependence." In *Dependence in Probability and Statistics*, 123–142. Springer.

Wu, W. B., and W. Min. 2005. "On Linear Processes With Dependent Innovations." *Stochastic Processes and their Applications* 115: 939–958.

Wu, W. B., and X. Shao. 2007. "A Limit Theorem for Quadratic Forms and Its Applications." *Econometric Theory* 23: 930–951.

Wu, W. B., and H. Xiao. 2012. "Covariance Matrix Estimation in Time Series." In *Handbook of Statistics*, edited by T. Subba Rao, S. Subba Rao, and C. R. Rao, 187–209. Elsevier.

Xiao, H., and W. B. Wu. 2012. "Covariance Matrix Estimation for Stationary Time Series." *Annals of Statistics* 40, no. 1: 466–493.

Zhang, D., and W. Wu. 2017. "Asymptotic Theory for Estimators of High-Order Statistics of Stationary Processes." *IEEE Transactions on Information Theory* 64: 4907–4922.

Zhang, X. 2016. "White Noise Testing and Model Diagnostic Checking for Functional Time Series." *Journal of Econometrics* 194: 76–95.

Zhou, Z. 2013. "Heteroscedasticity and Autocorrelation Robust Structural Change Detection." *Journal of the American Statistical Association* 108: 726–740.

Supporting Information

Additional supporting information can be found online in the Supporting Information section. **Data S1.** Supporting Information.