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NOTES AND PROBLEMS

IMPULSE RESPONSES OF FRACTIONALLY INTEGRATED PROCESSES WITH LONG MEMORY

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Fractionally integrated time series, which have become an important modeling tool over the last two decades, are obtained by applying the fractional filter $(1-L)^{-d} = \sum_{n=0}^{\infty} b_n L^n$ to a weakly dependent (short memory) sequence. Weakly dependent sequences are characterized by absolutely summable impulse response coefficients of their Wold representation. The weights b_n decay at the rate n^{d-1} and are not absolutely summable for long memory models ($d > 0$). It has been believed that this rate is inherited by the impulse responses of any long memory fractionally integrated model. We show that this conjecture does not hold in such generality, and we establish a simple necessary and sufficient condition for the rate n^{d-1} to be inherited by fractionally integrated processes.

1. PROBLEM AND MOTIVATION

This paper focuses on the asymptotic behavior of the impulse response coefficients of fractionally integrated sequences, which, since their introduction by Granger and Joyeux (1980) and Hosking (1981), have been extended in many directions and have become a useful modeling tool in geophysics, especially hydrology and meteorology, econometrics, computer science, and many other areas. See Doukhan, Oppenheim, and Taqqu (2003), Robinson (2003), and Teyssiere and Kirman (2006) for recent overviews.

Fractionally integrated processes are constructed by filtering a weakly dependent sequence with the filter $(1-L)^{-d} = \sum_{n=0}^{\infty} b_n L^n$, where L is the usual lag

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operator and

$$b_n = \frac{\Gamma(n+d)}{\Gamma(n+1)\,\Gamma(d)} = \frac{n-1+d}{n} b_{n-1}, \qquad n \geq 1, \qquad b_0 = 1. \tag{1.1}$$

The filter coefficients satisfy

$$b_n \sim \frac{n^{d-1}}{\Gamma(d)}, \qquad n \to \infty. \tag{1.2}$$

In equation (1.2), and in what follows, $a_n \sim \alpha_n$ denotes $a_n/\alpha_n \to 1$, as $n \to \infty$.

The preceding expansions and asymptotics are true for any noninteger d. If $d < 0.5$, the sequence $\{b_n\}$ is square summable. The range $0 < d < 0.5$ corresponds to stationary long memory processes.

Suppose that $\{e_t\}$ is a time series with the representation $e_t = \sum_{n=0}^{\infty} \psi_n \varepsilon_{t-n}$, in which the innovations ε_t are not necessarily independent and identically distributed and may not even have second moments. The filtered process has then the formal expansion

$$\begin{aligned} y_t &= (1-L)^{-d} e_t = \sum_{n=0}^{\infty} b_n e_{t-n} \\ &= \sum_{n=0}^{\infty} c_n \varepsilon_{t-n}, \qquad \text{with } 0 < d < 0.5 \end{aligned} \tag{1.3}$$

with the impulse responses given by the convolution

$$c_n = \sum_{k=0}^{n} \psi_k b_{n-k}, \qquad n \geq 0. \tag{1.4}$$

The idea of modeling using fractionally integrated processes is that the filter $\{\psi_k\}$ modifies the short memory dynamics, thus providing the desired flexibility, whereas the filter $\{b_k\}$ determines the large scale behavior of the model. For the popular fractionally integrated autoregressive moving average models, the $\{\psi_k\}$ decay exponentially fast, and it is indeed true that the c_n inherit the asymptotic behavior of the b_n, i.e., they decay at the rate n^{d-1}. This result was established in Kokoszka and Taqqu (1995), but it is implicit in the original work of Hosking (1981), who focuses on autocovariances rather than the impulse response coefficients.

There seems to be a common understanding that the c_n decay like n^{d-1} for general short memory filters $\{\psi_k\}$. See Baillie and Kapetanios (2008) for a recent example of such a view. Poskitt (2007, eqn. (5)), claims that the absolute summability of $\{\psi_k\}$ is sufficient. The purpose of this note is to clarify what condition must be imposed on the coefficients $\{\psi_k\}$ to ensure that the c_n decay like n^{d-1}. In particular, we show that the absolute summability of the $\{\psi_k\}$ is not sufficient, but an additional condition that is required is very weak and holds for all models of practical interest.

The asymptotics for the impulse response coefficients are useful in prediction problems (see, e.g., Pourahmadi, 2001), and in studying processes with heavy tails and/or conditional heteroskedasticity (see, e.g., Kokoszka and Taqqu, 1996; Kokoszka and Mikosch, 1997; Giraitis, Robinson, and Surgailis, 2000).

The remainder of the paper is organized as follows. Section 2 contains our results with some discussion, and Section 3 contains the proofs.

2. RESULTS AND DISCUSSION

Define the function

$$\Psi(x) = \psi_0 + \sum_{n=1}^{\infty} \psi_n e^{inx}$$

for those $x \in (-\pi, \pi)$ for which the series on the right-hand side converges. To establish our results, we need the following assumption.

Assumption 2.1. The function $\Psi(x)$ is continuous and of bounded variation on $(-\pi, \pi)$, and

$$\Psi(0) = \sum_{k=0}^{\infty} \psi_k \neq 0. \tag{2.1}$$

If the coefficients ψ_n satisfy

$$\sum_{n=0}^{\infty} |\psi_n| < \infty, \qquad \sum_{n=0}^{\infty} \psi_n \neq 0, \qquad \psi_0 = 1, \tag{2.2}$$

then Assumption 2.1 holds. Absolute summability is however not necessary. For example, Assumption 2.1 holds if $\psi_n = (-1)^n a_n$, where $a_n > 0$ decrease monotonically to zero; see e.g., Tolstov (1976, p. 103).

The main result of this paper is the following proposition.

PROPOSITION 2.1. *Suppose c_n is defined by (1.4) and (1.1) and Assumption 2.1 holds. Then, for* $0 < d < 1$*,*

$$c_n \sim \frac{\sum_{j=0}^{\infty} \psi_j}{\Gamma(d)} n^{d-1}, \qquad n \to \infty, \tag{2.3}$$

if and only if

$$n^{1-d} \psi_n \to 0, \qquad n \to \infty. \tag{2.4}$$

The proof of Proposition 2.1 is given in Section 3; the key relation is (3.2).

The following example shows that the absolute summability of the ψ_n does not imply condition (2.4). It also provides an example of an absolutely summable sequence $\{\psi_n\}$ for which (2.3) fails.

Example 2.1

Set $s = 1 - d$, so that $0 < s < 1$. We construct a sequence $\{\psi_n\}$ such that (2.2) holds and $n^s \psi_n \not\to 0$. Set $\psi_n = n^{-s} \mathbf{I}_A(n)$, $n > 1$, where $\mathbf{I}_A(n) = 1$ if $n \in A$ and

$\mathbf{I}_A(n) = 0$ if $n \notin A$. If the set A contains infinitely many positive integers, then $n^s \psi_n = \mathbf{I}_A(n) \not\to 0$.

Now, we must choose A so that $\sum_{n=1}^{\infty} n^{-s}\mathbf{I}_A(n) < \infty$. Set $A = \{n : n = \lceil k^\alpha \rceil, k = 1, 2, 3, \ldots\}$, where $\lceil x \rceil$ denotes the smallest integer not less than x and $\alpha > 1/s$. Because $\lceil k^\alpha \rceil \geq k^\alpha$, $\lceil k^\alpha \rceil^{-s} \leq k^{-\alpha s}$, and so $\alpha s > 1$ implies

$$\sum_{n=1}^{\infty} n^{-s}\mathbf{I}_A(n) = \sum_{k=1}^{\infty} \lceil k^\alpha \rceil^{-s} \leq \sum_{k=1}^{\infty} k^{-\alpha s} < \infty.$$

To show that (2.3) fails, we modify the set A slightly by setting $A^* = \{n : n = \lceil k^\alpha \rceil, k > k^*\}$, where k^* is so large that $\sum_{k > k^*} k^{-\alpha s} < \Gamma(d)$, and defining $\psi_n = n^{-s}\mathbf{I}_{A^*}(n)$. This implies that

$$\frac{\sum_{n=0}^{\infty} \psi_n}{\Gamma(d)} < 1. \tag{2.5}$$

Because the ψ_k and the b_k are nonnegative, we have, taking only the term with $k = n$ in (1.4), $n^{1-d} c_n \geq n^{1-d} \psi_n$. By the construction of ψ_n, we see that $n^{1-d} c_n \geq 1$, for infinitely many n. Thus, if relation (2.3) were true, we would have $\sum_{n=0}^{\infty} \psi_n / \Gamma(d) \geq 1$, but this contradicts (2.5).

Despite Example 2.1, we conclude that for most models of practical interest, the intuition that (2.3) holds under the absolute summability of the ψ_n is correct. A simple sufficient condition for (2.4) is $\limsup_{n\to\infty} n|\psi_n| < \infty$, which holds if $\psi_n \sim Cn^{-\gamma}$, $\gamma > 1$.

Formula (3.1) in the proof of Proposition 2.1 shows that

$$\limsup_{n\to\infty} n^{1-d} c_n = \frac{\sum_{j=0}^{\infty} \psi_j}{\Gamma(d)} + \limsup_{n\to\infty} (n^{1-d} \psi_n)$$

and

$$\liminf_{n\to\infty} n^{1-d} c_n = \frac{\sum_{j=0}^{\infty} \psi_j}{\Gamma(d)} + \liminf_{n\to\infty} (n^{1-d} \psi_n).$$

These relations can be used to impose conditions on the ψ_n that imply the rate n^{d-1} on the c_n without requiring that the limit of $n^{1-d} c_n$ is exactly $\sum_{j=0}^{\infty} \psi_j / \Gamma(d)$. By choosing the sequence ψ_n so that $\limsup_{n\to\infty} n^{1-d} \psi_n = \infty$ and $\liminf_{n\to\infty} n^{1-d} \psi_n = -\infty$, we see that that for absolutely summable ψ_n, the c_n do not have to decay at rate n^{d-1}.

For ease of reference we state the following corollary, which describes the asymptotic behavior of autocovariances of the fractionally integrated process $\{y_t\}$ given by (1.3). It follows from Phillips (2010), which shows that even under more general conditions (see in particular Phillips, 2010, eqn. (21)) (2.3) implies a hyperbolical decay of the autocovariances.

COROLLARY 2.1. *Suppose* $e_t = \sum_{n=0}^{\infty} \psi_n \varepsilon_{t-n}$, *where* $\{\varepsilon_t\}$ *is a mean zero white noise sequence with variance* σ^2. *If conditions (1.1), (1.4), (2.2), and (2.4) hold and* $0 < d < 1/2$, *then the autocovariances* $\gamma(h) = \mathrm{E}(y_t y_{t+h})$ *of the fractionally integrated process (1.3) satisfy*

$$\gamma(h) \sim h^{2d-1}\sigma^2 \left(\sum_{j=0}^{\infty} \psi_j\right)^2 \Gamma(1-2d)\frac{\sin(\pi d)}{\pi}, \quad \text{as } h \to \infty. \tag{2.6}$$

The restriction to $0 < d < 1/2$ in Corollary 2.1 was needed to consider autocovariances that are defined for stationary processes. Proposition 2.1 applies however to $0 < d < 1$, and therefore we may also consider impulse response coefficients for the nonstationary case with $1/2 \le d < 1$. To that end consider the so-called fractionally integrated processes $\{z_t\}$ of "type II" (see Marinucci and Robinson, 1999; Robinson, 2005), which are defined by

$$z_t = \sum_{n=0}^{t-1} b_n e_{t-n} = \sum_{n=0}^{\infty} a_n \varepsilon_{t-n}, \qquad 0 < d < 1, \tag{2.7}$$

where $\{b_n\}$ is from (1.1) and $\{a_n\}$ is given again by convolution. The impulse responses a_n coincide with the c_n from (1.4) for $n \in \{1, \ldots, t-1\}$; for $n \ge t$, however, $a_n = \sum_{k=0}^{t-1} \psi_{n-k} b_k$. Note that the sequence $\{z_t\}$ is nonstationary for $1/2 \le d < 1$ in that $\sum_n b_n^2$ does not converge. But even for $d < 0.5$, $\{z_t\}$ forms a nonstationary sequence; the difference between the stationary "type I" process $\{y_t\}$ in (1.3) and the type II counterpart is $y_t - z_t = \sum_{n=t}^{\infty} b_n e_{t-n}$ with $\mathrm{E}(y_t - z_t)^2 = O(t^{2d-1})$; see eqn. (1.6) in Robinson (2005).

3. PROOF OF PROPOSITION 2.1

Consider the functions

$$B(x) = b_0 + \sum_{n=1}^{\infty} b_n e^{inx} \quad \text{and} \quad \Psi(x) = \psi_0 + \sum_{n=1}^{\infty} \psi_n e^{inx}.$$

Recall that $B(x) = (1 - e^{ix})^{-d}$ and the function $\Psi(x)$ is well defined by Assumption 2.1. Define

$$C(x) = B(x)\Psi(x)$$

and its Fourier series

$$C(x) \sim c_0 + \sum_{n=1}^{\infty} c_n e^{inx}.$$

The argument that follows uses the results of Section 2 in Chapter 5 of Zygmund (1959), which show that the Fourier transform of a regularly varying sequence is a regularly varying function as the frequencies approach zero, and vice versa. In our case, the slowly varying functions are asymptotically constant, and so the exact power rates are preserved. The calculations that follow are valid for $0 < d < 1$. By $L_1(x), L_2(x)$ we denote slowly varying functions, and by $K_1(d), K_2(d)$ we denote constants whose values are not important for the proof.

First we represent b_n as $b_n = n^{d-1}L_1(n)$. Because $L_1(n) \to [\Gamma(d)]^{-1}$, as $n \to \infty$, the function $L_1(\cdot)$ is slowly varying at infinity. Thus, by Theorem 2.6 on p. 187 of Zygmund (1959) $B(x) - b_0 \sim x^{-d}L_1(x^{-1})K_1(d),\ x \to 0+$. This means that

$$B(x) - b_0 = x^{-d}L_1(x^{-1})K_1(d)L_2(x),$$

where $\lim_{x\to 0+} L_2(x) = 1$ (so, in particular, $L_2(\cdot)$ is slowly varying as $x \to 0+$).

Therefore

$$C(x) = \left(b_0 + x^{-d}L_1(x^{-1})K_1(d)L_2(x)\right)\Psi(x),$$

and so

$$C(x) - b_0\Psi(x) = x^{-d}L_1(x^{-1})K_1(d)L_2(x)\Psi(x).$$

The function $C(x) - b_0\Psi(x)$ is thus regularly varying as $x \to 0+$ because $L_1(x^{-1})K_1(d)L_2(x)\Psi(x)$ tends to a constant. It has Fourier coefficients $c_n - b_0\psi_n$ and so, by Theorem 2.24 on p. 190 of Zygmund (1959),

$$c_n - b_0\psi_n \sim n^{d-1}L_1(n)\Psi(n^{-1})K_2(d)L_2(n^{-1}).$$

Hence, by (2.1), $c_n - b_0\psi_n \sim n^{d-1}[\Gamma(d)]^{-1}\left(\sum_{k=0}^{\infty}\psi_k\right)K_2(d)$. To find the constant $K_2(d)$ notice that it does not depend on the sequence $\{\psi_n\}$, and so to determine it we examine the preceding relation with a simple sequence $\{\psi_n\}$ defined by setting $\psi_0 = 1$ and $\psi_n = 0, n \geq 1$, so that $c_n = b_n$ and $b_0\psi_n = 0,\ n \geq 1$. Thus, (1.2) implies that $K_2(d) = 1$. We conclude that for any sequence $\{\psi_n\}$ satisfying (2.1)

$$c_n - b_0\psi_n \sim n^{d-1}[\Gamma(d)]^{-1}\Psi(0). \tag{3.1}$$

Because $b_0 = 1$, relation (3.1) can be rewritten as

$$n^{1-d}c_n - n^{1-d}\psi_n \to \frac{\Psi(0)}{\Gamma(d)}, \tag{3.2}$$

and so the equivalence (under (2.1)) of (2.3) and (2.4) follows. ■

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