STAT 540, Fall 2015

Due on **12:00 A.M., Friday, Oct 16th**

Homework 6

Homework format for all STAT 540 homework this term: Please label all problems clearly and turn in an organized homework assignment. You don't need to spend hours producing beautifully typeset homework, but you won't get credit if we can't find or read your answer. Unless noted otherwise, turn in the following (as appropriate for the problem).

- Theoretical derivation (when asked for).
- Numerical results **with an explanation of your solution**, written in complete sentences. If computer code is absolutely necessary to provide context here, then include it–nicely formatted–within the solution (otherwise, see below).
- Appropriate graphics. Use informative labels, including titles and axis labels. Try to put multiple plots on the page by using, for example, the R command `par(mfrow=c(2,2))`.
- **Only as necessary**: Final clean computer code used to answer the problem **attached to the end of your homework**. Only include the rare code excerpts without which we wouldn't be able to figure out what you did. Annotate your code. Number and order the code in order of the problems. When in doubt, leave it out; consider that we will probably never read it.
- Some problems will be relatively open-ended, such as "Here are some data. Analyze them and write a report." I will provide further instructions about reports later. They should be self-contained, with suitable EDA, graphs, numerical results, and **scientific interpretation**. No computer code should be included. The report should be concise: "no longer than necessary".

(1) Basic review on matrix operations. Define

$$\mathbf{X} = \begin{pmatrix} 1 & 6 \\ 1 & 5 \\ 1 & 2 \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \ \mathbf{z} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Evaluate the following quantities, if possible. Indicate when an operation cannot be performed.

(a) $\mathbf{y} + \mathbf{X}$
(b) $\mathbf{y} - \mathbf{z}$
(c) $\mathbf{y}'\mathbf{z}$

(d) $\mathbf{y}\mathbf{z}'$
(e) $\mathbf{A} + \mathbf{C}$
(f) $\text{trace}(\mathbf{C})$
(g) $\det(\mathbf{C})$
(h) $\mathbf{C}^{-1}$
(i) $\mathbf{A}^{-1}$
(j) $\mathbf{A}^{-1}\mathbf{A}$
(k) $\mathbf{C}\mathbf{X}$
(l) $\mathbf{X}\mathbf{C}$
(m) $\mathbf{X}'\mathbf{X}$
(n) $(\mathbf{X}'\mathbf{X})^{-1}$
(o) $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}\mathbf{y}$
(p) $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$
(q) $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$
(r) $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$

(2) Review of matrix algebra and applications in statistics.

(a) Denote $\mathbf{H}$ the hat matrix in the lecture note (recall yourself what is $\mathbf{H}$ in terms of the design matrix $\mathbf{X}$). Show that

(i) $\mathbf{H}$ is symmetric, and
(ii) $\mathbf{H}$ is idempotent, that is $\mathbf{H}\mathbf{H} = \mathbf{H}$.

(b) The following questions are about the rank of matrix, particularly, the design matrices.

(i) Determine the rank of the following matrix

$$\begin{pmatrix} 3 & 2 & 7 & 1 & 5 \\ -1 & 3 & 5 & 1 & 3 \\ 4 & 2 & 8 & 7 & -6 \end{pmatrix}$$

(ii) Consider the design matrix on slide 14 of the lecture note 7. What is the largest possible rank of this matrix, and what must be true about $x_1, \dots, x_8$ to achieve that maximum rank?

(iii) Consider the design matrix on slide 25 of the lecture note 7.

I. Determine the rank of the design matrix.
II. The column space of a matrix is the set of vectors spanned by the columns of the matrix. Give another matrix with the same column space as the design matrix

(c) Let $\mathbf{A}$ be matrices of real numbers, show that $\mathbf{A} = \mathbf{0} \Leftrightarrow \text{trace}(\mathbf{A}'\mathbf{A}) = \mathbf{0}$.

(d) Suppose $a_1, \dots, a_4 \in \mathbb{R}$. Consider matrix

$$\mathbf{A} = \text{diag}(a_1, a_2, a_3, a_4) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

(i) Find the determinant of $\mathbf{A}$.

(ii) Find the eigenvalues of $\mathbf{A}$.

(e) Find a so that $[6\ 1\ \ -2]'$ is the span of

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ 8 \end{bmatrix} \right\}.$$

(3) Review of multivariate random variables.

(a) If $\mathbf{x}, \mathbf{y}$ are random vectors and $\text{Cov}(\mathbf{x}, \mathbf{y})$ is defined as $\mathbb{E}[\{\mathbf{x} - \mathbb{E}(\mathbf{x})\}\{\mathbf{y} - \mathbb{E}(\mathbf{y})\}']$. Suppose $\mathbf{A}$ and $\mathbf{B}$ are matrices of constants, $\mathbf{a}$ and $\mathbf{b}$ are vectors of constants, and $\mathbf{z}$ is a random vector with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\Sigma}$. Give a simplified expression for $\text{Cov}(\mathbf{Az} + \mathbf{a}, \mathbf{Bz} + \mathbf{b})$.

(b) Suppose W_1 and W_2 are random variables. Suppose that $\text{Var}(W_1) = 4, \text{Var}(W_2) = 2$ and $\text{Cov}(W_1, W_2) = -1$. Find out

$$\text{Var}\left(\begin{bmatrix} W_1 + W_2 \\ W_1 - W_2 \end{bmatrix} \right).$$

(c) Assume that $y_1, \ldots, y_n \sim N(\mu, \sigma^2)$ are i.i.d. random variables. Denote $\mathbf{y} = [y_1, \ldots, y_n]'$, and let $\bar{y} = n^{-1} \sum_{i=1}^n y_i$.

(i) Show that $s^2 = (n-1)^{-1} \sum_{i=1}^n (y_i - \bar{y})^2$ can be written as $\mathbf{y}'\mathbf{B}\mathbf{y}$ for some matrix $\mathbf{B}$.

(ii) Prove that $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$ using what you have learned from Stat 520 or other elementary probability courses.

(4) Consider the simple linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i,\ i = 1, \ldots, n$$

where $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. $N(0, \sigma^2)$ and β_0, β_1 and $\sigma > 0$ are unknown parameters. The model matrix for this linear regression model can be written as $\mathbf{X} = [\mathbf{1}, \mathbf{x}]$ as we did in class.

(a) We have learned that the least squares estimator of $\boldsymbol{\beta}$ in a general linear model with a full rank matrix $\mathbf{X}$ is given by $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$. Simplify this expression for the special case of simple linear regression to obtain expression for the least squares estimators of β_0 and β_1. Express your final answers using summation notation.

(b) There are some computational advantages to working with a design matrix whose columns are orthogonal. For the simple linear regression problem consider the design matrix

$$\mathbf{W} = [\mathbf{1}, \mathbf{x} - \bar{x}\mathbf{1}].$$

This design matrix is obtained by centering the explanatory variable around its mean $\bar{x} = n^{-1} \sum_{i=1}^n x_i$. Find a matrix $\mathbf{B}$ so that $\mathbf{X}\mathbf{B}^{-1} = \mathbf{W}$. It will follow that

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{X}\mathbf{B}^{-1}\mathbf{B}\boldsymbol{\beta} = \mathbf{W}\boldsymbol{\alpha}$$

where $\mathbf{B}\boldsymbol{\beta} = \boldsymbol{\alpha}$.

(c) Derive expressions for the least squares estimators of α_0 and α_1, where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1)'$ from part (b)) using $\hat{\boldsymbol{\alpha}} = (\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{y}$.

(d) Multiply $\hat{\boldsymbol{\alpha}}$ from part (c) by $\mathbf{B}^{-1}$ from part (b) to obtain expression for $\hat{\beta}_0$ and $\hat{\beta}_1$.

(e) Show that your answer to part (a) matches your answer to part (d).

(5) Textbook problems:

(a) Problem 5.3, 5.17

(b) Problem 6.15: Patient satisfactions. Parts (a)-(e) only. In (a), replace stem-and-leaf plots with histograms; also use studentized residuals (`studres(mymodel)` in `R`.)

(c) Problem 6.16 Patient satisfactions continued.